The University of Dublin

School of Mathematics

Quantum Aspects of Black Holes in de Sitter Spacetime

Thomas P. Brosnan

Supervised by Prof. Manuela Kulaxizi

April 6th, 2025

Submitted in partial fulfillment of the requirements for the degree of
B.A. (Hon.) Theoretical Physics

Abstract

Developing a successful theory of quantum gravity is one of the most complex challenges facing theoretical physicists today. Some of the greatest jumps in our understanding of how gravity and quantum mechanics interact, have come from studies of Quantum Field Theory (QFT) in curved spacetime. This project studied how quantum fields propagating in different space-times can lead to a plethora of intriguing results, particularly with regards to black holes. First the Unruh effect [18] was studied to introduce how QFT can lead to particle creation for accelerated observers. Next, building on this, Hawking's original paper [13] on Hawking radiation was analysed to see how this could be extended to black holes. Finally Raphael Bousso and Stephen Hawking's 1997 paper "(Anti)Evaporation of Schwarzchild-de Sitter Black Holes" [3] was studied to see how a Schwarzchild black hole immersed in a de Sitter background can evolve in the presence of quantum fields, leading to interesting dynamics.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Professor Manuela Kulaxizi, for her wisdom and guidance throughout this project. Her insight and advice have been instrumental in deepening my understanding of both the technical details of the project and its broader significance. I would also like to thank my friends and family for their support, particularly Sean for his proofreading of my report and my Mom for her endless patience. Lastly, I am deeply grateful to my girlfriend, Ellie, for her unwavering love and support, as well as her careful review of my report. I would be unintelligible without her.

I have read and understood the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar. I have also read and understood the guide, and completed the 'Ready Steady Write' Tutorial on avoiding plagiarism, located at https://libguides.tcd.ie/ac ademic-integrity/ready-steady-write. Thomas Brosnan 6th April 2025

"Consideration of particle emission from black holes would seem to suggest that God not only plays dice, but also sometimes throws them where they cannot be seen."

-*Stephen Hawking*

Contents

Chapter 1

Introduction & Background

The two pillars on which, most, if not all of modern theoretical physics stand on, are General Relativity (GR) and Quantum Field Theory (QFT). On one hand these theories are so fundamentally different, that we have so far, been unsuccessful at creating a theory of quantum gravity. On the other hand, these are both field theories and hence are somewhat compatible at a semi-classical level. By studying instances where the effects of both quantum fields and gravity are present and relevant, but not in a way that spacetime changes on a large scale; we can learn a lot about how these two different theories interact. This type of analysis is important as it cannot only lead to new and interesting physics, but it also gives us a outline, in the form of semi-classical limits, for what an eventual theory of quantum gravity will look like. Studying the propagation of quantum fields on a fixed, but curved background is possible and is known simply as QFT in curved spacetime, the framework of which describes physics at low energies, far below the Planck scale.

The first promising results that displayed that QFT in curved spacetime could lead to new physics came in the early 1970's. In 1973, Jacob Bekenstein [1] proposed that Black holes have entropy proportional to the area of their horizons. In an attempt to explain how this could fit into the picture of thermodynamics Stephen W. Hawking in 1975 [13] applied QFT to the space time surrounding a black hole, showing that they can in fact radiate as if they have a temperature. In 1976, a year later, William G. Unruh [18] showed that radiation could even be seen in flat time for an observer accelerating through a vacuum.

By the late 1990's many further techniques had developed for explaining and encoding evaporation and other quantum effects into models involving gravity. In 1997 Raphael Bousso and Stephen W. Hawking [3] used some of these techniques to study the evolution of a Schwarzschild black hole immersed in a de Sitter background.

The aim of this work is to gather and explore a selection of the results associated with QFT in curved spacetime with a view to presenting them as a sort of technical history of the concept, the structure of which is as follows. In the rest of this Chapter 1, a brief introduction to QFT and GR is given as well as de Sitter spacetime. In Chapter 2, the Unruh effect is investigated to introduce how temperature can arise from QFT, before using this as a base to understand Hawking's original paper in Chapter 3. Finally Chapter 4 takes a look at Raphael Bousso and Stephen Hawking's 1997 paper "(Anti-)Evaporation of Schwarzschild-de Sitter Black Holes".

1.1 Quantum Field Theory

Quantum field theory is the natural extension of quantum mechanics to incorporate special relativity. In quantum mechanics, quantities such as position are promoted to operators in order to represent how measurements can affect the states they are acting on. However, time is not given such treatment and is instead left as a background parameter. Einstein's theory of special relativity tells us that we must treat space and time in the same way, contradicting this very notion. Clearly to reconcile these two theories, something in quantum mechanics needs to change. There are two ways of going about this. The first approach would be to promote time to an operator, but this gets messy quite quickly. The other is to demote position x back to being a parameter. But what are the consequences of this? It just means that what ever object we are using to describe our system with must be a function of the continuous variable x and t. This is identical to how a field is described and this fact forms the motivation for quantum field theory.

1.1.1 Klein Gordon Field

Once we have established that the fundamental objects needed to describe relativistic quantum physics are fields $\phi(\mathbf{x}, t)$[1], how do we proceed? The natural progression is to impose that these fields must obey the relativistic energy equation. That is, they must satisfy $E^2 = \mathbf{p}^2 + m^2$. But this is not in the language of quantum mechanics. To do that we need to change $\mathbf{p} \to -i\hbar\nabla$ and $E \to i\hbar\partial_t$. Applying this version of the relativistic energy equation to our field ϕ gives us (in four-notation[2]):

$$\left(\partial_t^2 - \nabla^2\right)\phi(\mathbf{x}, t) = \partial_\mu\partial^\mu\phi = m^2\phi \tag{1.1}$$

$$\implies \left(\partial_\mu\partial^\mu - m^2\right)\phi = 0 \tag{1.2}$$

This is whats is know as the *Klein Gordon Equation* and the fields that satisfy this are the simplest fields one can study in QFT. Particles whose fields satisfy this equation turn out to be spin 0 particles.

Solutions to the Klein Gordon Equation

Throughout this work, calculations can be made less complex by studying a massless spin 0 Klein Gordon field, which will satisfy $\partial_\mu\partial^\mu\phi \equiv \Box\phi(x) = 0$. This equation can be solved by assuming a separable ansatz, namely $\phi(\mathbf{x}, t) = \chi(\mathbf{x})f(t)$. We can then plug this ansatz into the KG equation 1.1 and divide across by $-\phi = -\chi\phi$:

$$-\frac{1}{f}\partial_t^2 f + \frac{1}{\chi}\nabla^2\chi = m^2$$

Since $\mathbf{x}$ and t are independent, both the terms on the LHS of this equation must be equal to constants. Thus we suggestively write:

$$-\frac{1}{f}\partial_t^2 f = E^2 \implies f \propto e^{\pm iEt}$$

$$\frac{1}{\chi}\nabla^2\chi = -\mathbf{p}^2 \implies \chi \propto e^{\pm i\mathbf{p}\cdot\mathbf{x}}$$

[1]Note that we have assumed here that our fields are scalar fields, meaning they just have a single number attributed to them at each point in spacetime.

[2]Note in this work we use natural units $\hbar = G = k = 1$ and the mostly plus Minkowski metric $(- + + +)$.

Through imposing that the field ϕ must evolve through the standard unitary time evolution operator of quantum mechanics, one finds that in the combination $\phi = \chi f$, the exponents $\pm iEt$ and $\pm i\mathbf{p}\cdot\mathbf{x}$ must be of opposite sign. Hence, there are two types of modes for the KG to be made up of. Following proper ODE methods, one can write down the most general combination as a linear superposition of all possible values of the constants E and $\mathbf{p}$. Since they are related via the equation $E^2 = \mathbf{p}^2 + m^2$, which we can now recognize as the energy equation, we only need to sum over one of these parameters. Also, seeing as they can take on any values, the superposition should be an integral, with the coefficients of the modes depending on $\mathbf{p}$[3].

$$\phi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot x}\right] \tag{1.3}$$

Notice here that we have also defined the coefficients of the modes with an extra factor of $1/\sqrt{2E_{\mathbf{p}}}$, where $E^2_{\mathbf{p}} = \mathbf{p}^2 + m^2$. This is added so that this field is correctly normalized with respect to the KG inner product, which we introduce in section A.1.4. We also have that the coefficients of the two modes are related, since we have a and $a^\dagger$. This is because these coefficients can be complex but must take this form for the entire integral to be real, which we require for a real scalar field.

1.1.2 Second Quantization

To quantize the above presented field theory, ϕ needs to be promoted to an operator that satisfies a set of commutation relations. We have not written down a Lagrange density yet, but it is straightforward to show that it takes the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{1.4}$$

The equations of motion of the Lagrange density Klein Gordon equation. With a Lagrangian defined it is then sensible to interpret ϕ as playing the role of a position co-ordinate, meaning its conjugate quantity, which we will call the *momentum density*, is defined as:

$$\pi(\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\mathbf{x})}$$

For the above defined Lagrangian 1.4 we can see that $\pi = \dot{\phi}$ and hence π takes the following form according to 1.3:

$$\pi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}}e^{-ip\cdot x} - a^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot x}\right] \tag{1.5}$$

With a conjugate quantity defined, that which is left to do is to define commutation relations such that $[\phi(x), \pi(y)] = i\delta(\mathbf{x}-\mathbf{y})$. If one plugs 1.3 and 1.5 above into this condition[4], then it is equivalent to the coefficients $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$, which are now operators in the quantised theory, satisfying the relations:

$$\begin{aligned}\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] &= (2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \\ \left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] &= \left[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = 0\end{aligned} \tag{1.6}$$

Making them creation and annihilation operators!

[3]Note here we switch to 4-vector notation and hence must recognize p^0 as $E_{\mathbf{p}}$.
[4]For proof of this as well as a broader overview of QFT see these notes I made [4].

1.2 General Relativity

General relativity is the extension of Einstein's theory of special relativity to include gravity. It continues to treat space and time on the same footing, but its main difference to special relativity is that it allows time and space to "bend", a drastically different approach to Newtons idea of a fixed and absolute space. More formally, space and time together are treated as a manifold with the curvature and bending of space encoded on this manifold in form of a metric. This metric tells us how close together any two vectors defined at the same point are to each other. It must be a smooth function so that these angles vary continuously as we move from point to point. This in turn fixes the curvature of space as the tangent vectors of particles on trajectories change smoothly from point to point, tracing out a path in spacetime that may no longer be straight. This description of space and time allows gravity to be explained in a very elegant way. We simply require that the curvature of spacetime is *caused* by the presence of mass. Then the particles moving in the presence of massive objects follow curved paths, not because they are experiencing a force, but because they are following "straight" lines on the curved manifold of spacetime. These lines are of course no longer straight, instead we classify them by the curves that are parallel transported by their own tangent vectors. These paths are known as *geodesics*.

1.2.1 Tensors

There is another issue that needs to addressed in the formalism of general relativity. This theory should, as in special relativity, not be dependent on any one set of co-ordinates to define our physics with respect too. We should instead be able to make any co-ordinate transformation and arrive at the same results. This can be achieved by imposing that the objects we use are tensors (For example the metric we mentioned earlier is a tensor denoted $g_{\mu\nu}$). These are defined by their transformations that leave the total tensor invariant as the components and basis elements transform in opposite ways. The components of an n-tensor transform under a change of co-ordinates from the system x^μ to y^ν via:

$$T^{\mu_1 \cdots \mu_n} \to \frac{\partial x^{\mu_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\mu_n}}{\partial y^{\nu_n}} T^{\nu_1 \cdots \nu_n}$$

This independence on co-ordinate system makes general relativity a background independent theory.

1.2.2 Christoffel Symbols

As in many theories we would like to be able to see how some of our parameters, such as the components of vectors, change with respect to our co-ordinates. However, the partial derivative of the component of a vector: $\partial_\mu V^\nu$ does not transform like a tensor, since when $V^\mu \to \frac{\partial x^\mu}{\partial x^\nu} V^\nu$, the factor of $\frac{\partial x^\mu}{\partial x^\nu}$ may depend on co-ordinates and hence the derivative will hit this creating an extra term through the product rule. The best way to fix this is to define a so called *co-variant derivative* which is the partial derivative plus a term that will exactly cancel this extra term from the product rule. Consistently defining the derivative this way leads to the new following action on the components of a vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma$$

Where we have defined the *Christoffel symbols* $\Gamma^{\nu}_{\mu\sigma}$. The co-variant derivatives operation on higher order tensors will involve more of these symbols as there are more of the product rule terms to cancel. Notably the action of the co-variant derivative on a scalar is just the partial derivative as a scalar does not transform. In terms of the metric tensor $g_{\mu\nu}$ it turns out that these symbols are given by:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu} \right)$$

1.2.3 Riemann Curvature

To examine the structure of curved spacetime it is useful to attempt to describe how much it "curves". Curvature should classify how different a space is to being flat. One of the main properties of curved space, that we may take for granted, is that the order of operations of things does not matter. We can go this way and then that way, or that way and then this way and end up at the same spot no matter what. If we note that the co-variant derivative of a vector in a direction along which it is parallel transported is zero, then we can use the co-variant derivative as a measure how much the path of a vector changes relative to how it would have been parallel transported. The natural extension is to then consider the difference between between applying the co-variant derivative in one direction then another, vs the other way round. This leads us to define the following quantity:

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} \equiv R^{\rho}{}_{\lambda\mu\nu} V^{\lambda}$$

This tensor is known as the *Riemann curvature tensor* and encodes with in it everything we need to know about the curvature of a space-time. If we evaluate out this commutator we find that in terms of the Christoffel symbols:

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_{\mu} \Gamma^{\rho}_{\nu\lambda} - \partial_{\nu} \Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\nu\sigma} \Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\rho}_{\nu\sigma} \Gamma^{\sigma}_{\mu\lambda}$$

Having a tensor with 4 indices is not always useful so we often consider contractions of this tensor. The first is known as the *Ricci tensor*:

$$R_{\mu\nu} = R^{\lambda}{}_{\lambda\mu\nu}$$

And further more we can contract this with the metric to get a scalar:

$$R = g^{\mu\nu} R_{\mu\nu}$$

This is know as the *Ricci curvature.*

There are not many scalar quantities we can think of to possibly use to write down an action principle in GR. The contractions of the metric with itself just give us the dimension of the spacetime, and we can always find co-ordinates such that the first derivatives of the metric vanish so the first non trivial action must be made of a scalar made out of second derivatives of the metric. This is exactly what the Ricci scalar is. It turns out that doing this works out exactly leading us to the Einstein Hilbert action, which we will see later.

1.2.4 Schwarzschild Black Holes

According to Birkhoff's theorem (See pg 197 of [6] for proof), the Schwarzschild metric is the unique, spherically symmetric solution to the vacuum Einstein equation: $R_{\mu\nu} = 0$. The Schwarzschild metric takes the following form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad f(r) = \left(1 - \frac{2M}{r}\right) \tag{1.7}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We will try, where possible, to carry out the calculations using $f(r)$ such that the results may be easily generalized to other potentials of the same form, but with different $f(r)$. However, when specifically talking about Schwarzschild black holes, we must have that $f(r) = \left(1 - \frac{2M}{r}\right)$ as above. This solution does not specify that the source itself be static. For example we could have the case of a collapsing star, and as long as the collapse is spherically symmetric, the metric can still be of the form 1.7. This will be the scenario that will be considering later in the chapter on Hawking Radiation 3.

1.2.5 Singularities

From the above metric, it can be seen 1.7 that there is a co-ordinate singularity, in that, as $r \to 2M$ we have $f(r) \to 0$, so the metric becomes undefined. We will see later that this is just due to our choice of co-ordinates, and that there is a co-ordinate system in which the metric is well behaved. There is also clearly a co-ordinate singularity at $r = 0$. This one, however, cannot be tamed by any change of co-ordinates. In fact there are co-variant quantities, such as:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$$

These clearly blow up as $r \to 0$, meaning $r = 0$ is a singularity. Hence, GR cannot accurately describe what is happening at this point.

1.2.6 Null Geodesics

To best determine both the causal structure of spacetime and the paths followed by massless scalar particles that make up the radiation seen later in this work, it is worth studying null geodesics. Null geodesics satisfy $ds^2 = 0$, so by fixing the angles θ and ϕ, $d\Omega = 0$ and the metric 1.7 as:

$$\begin{aligned} 0 = ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} \\ \implies \frac{dt}{dr} &= \pm\frac{1}{f(r)} = \pm\left(1 - \frac{2M}{r}\right)^{-1} \end{aligned} \tag{1.8}$$

From this, it can be extrapolated that a massless particle on a geodesic that is traveling towards the origin, as measured by a distant observer who is using the t time co-ordinate, never seems to get there. This is because if we integrate 1.8 to solve for t we find (see appendix B.1.1 for integration):

$$t - t_0 = r + 2M\ln\left(\frac{r}{2M} - 1\right) \tag{1.9}$$

From this it is clear that the total time for a massless particle to fall into the black hole diverges. Furthermore, since $\frac{dr}{dt} \to 0$ as $r \to 2M$, progress along a null geodesic gets slower and slower as the massless particle travels into the black hole. Therefore, we need different co-ordinates to make sense of this. By instead parametrising the geodesics by the time taken to fall in, we can use our expression from 1.9 to define the so called *Tortoise co-ordinate* r^* where:

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{1.10}$$

With this, the issue of the singularity at $r = 2M$ is lessened by placing it at infinity. In these co-ordinates the metric takes the below, form using the fact that $dr^* = dr/\left(1 - 2M/r\right)$:

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left[-dt^2 + dr^{*2}\right] + r^2 d\Omega^2 \tag{1.11}$$

Where here r should be thought of as a function of r^*. If we want to be more general, to get any $f(r)$ metric to this form, we would require that $f(r)dr^{*2} = \frac{1}{f(r)}dr^2 \implies r^* = \pm \int \frac{1}{f(r)}dr + C$. Which has the metric $f(r)\left[-dt^2 + dr^{*2}\right] + r^2 d\Omega^2$.

1.2.7 In/Out-going Geodesics

We can see from this metric 1.11 that null geodesics now just have $-dt^2 + dr^{*2} = 0 \implies dt = \pm dr^*$, which means $t \mp r^* = \text{const}$. So we have two co-ordinates that characterize the geodesics of massless particles [5]:

$$u = t - r^*$$
$$v = t + r^*$$

These are known as *Eddington-Finkelstein co-ordinates.* To see what is the difference between these co-ordinates, notice that for u we have:

$$dt = dr^* \implies dt = \frac{dr}{1 - \frac{2M}{r}} \implies \frac{dr}{dt} = 1 - \frac{2M}{r}$$

Then seeing as these co-ordinates only make sense for $r \geq 2M$, we must have that $\frac{dr}{dt} \geq 0$. The radius increasing in time means, these are *outgoing solutions.* For v we have the opposite as $dt = -dr^*$, meaning they correspond to *ingoing solutions.* We can note that the metric in these co-ordinates takes the following form for v and r (see appendix B.1.2):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2 \tag{1.12}$$

Or in just u and v:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv + r^2 d\Omega^2 \tag{1.13}$$

These co-ordinates correspond to the black hole equivalent of left and right moving modes and we will later split our scalar field into this modes as we did for the left and right moving in 2.29.

[5] Note that we have used the same notation for these co-ordinates as we did for the light-cone co-ordinates in the last chapter. This is because if we take the limit as $r >> 2M$, we can see that 1.10 reduces to $r^* = r$. So we end up with the same co-ordinates far from the black hole.

1.2.8 Kruskal co-ordinates

To talk about co-ordinates that are regular at the horizon, it is essential to "bring back" the horizon at $r = 2M$ from being infinitely far away, to having a finite value. ideal choices for these are:

$$\begin{aligned} U &= -e^{-u/4M} \\ V &= e^{v/4M} \end{aligned} \tag{1.14}$$

These are colloquially referred to as *Kruskal-Szekeres co-ordinates*, but technically these refer to $T = \frac{1}{2}(V + U)$ and $R = \frac{1}{2}(V - U)$, which are the corresponding time-like and space-like co-ordinates respectively. This work however, will still often refer to U and V as Kruskal co-ordinates. The metric in these co-ordinates takes the following form (see appendix B.1.3):

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2d\Omega^2 \tag{1.15}$$

Notice that this metric does not have a singularity at $r = 2M$, which means $r = 2M$ is just a co-ordinate singularity as previously mentioned. This also means that these co-ordinates are suitable candidates for co-ordinates near $r = 2M$, but a more rigorous argument for this will be made later in Chapter 3.

1.2.9 Penrose Diagrams

Most of the above mentioned co-ordinate transformations are quite useful in describing different sections and properties of black holes. However, with the horizon $r = 2M$, either being a singular point of the metric, or infinitely far away it is difficult to visualize what spacetime looks like near the black hole. Kruskal co-ordinates *almost* solve this problem except they do not fit all of spacetime on a single diagram. To achieve this it is necessary to use a function such as arctan that takes infinities to finite values. We also want to make sure to compactify along the light-cones, hence removing the infinities in the light cone co-ordinates. This results in the preservation of light-cone angles, meaning massless particles still travel at 45° in our new diagrams as the metric will become conformally flat (ignoring angular co-ordinates). Let us thus posit the following variable:

$$\begin{aligned} u' &= \arctan(U) \\ v' &= \arctan(V) \end{aligned} \tag{1.16}$$

With these the metric takes the form:

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}\frac{du'dv'}{\cos^2 u' \cos^2 v'} + r^2d\Omega^2 \tag{1.17}$$

With u' and v' it is then possible to separate them out into time-like and space-like co-ordinates in the usual manner. $\tau = \frac{1}{2}(u' + v')$ and $\rho = \frac{1}{2}(v' - u')$. Figure 1.1 shows a plot of these diagram of these τ and ρ co-ordinates along with lines of constant .

1.3 De Sitter spacetime

De Sitter spacetime is a maximally symmetric solution to Einstein's gravitational field equations involving a positive cosmological constant, which results in constant positive curvature. In physical terms de Sitter space describes a universe undergoing accelerated expansion, which is known from observations to be the case for our universe.

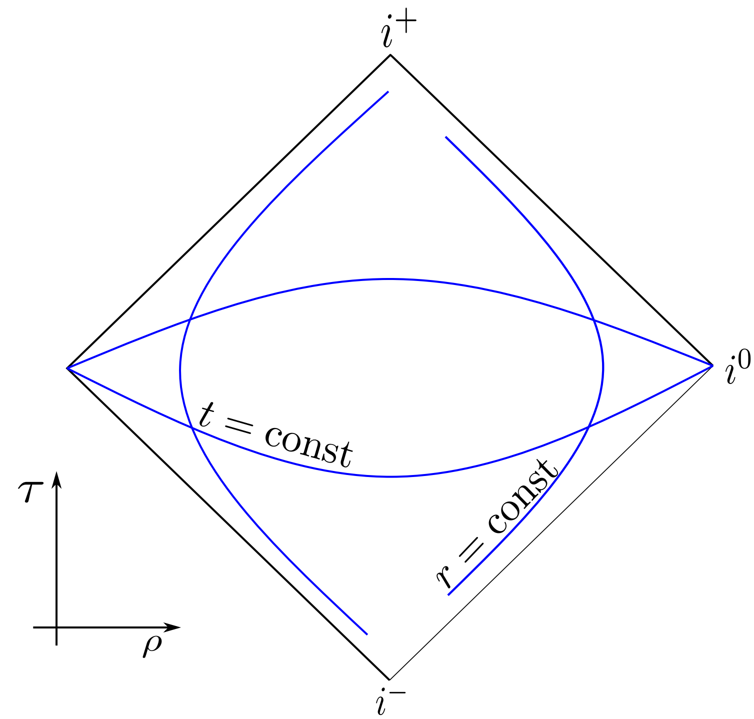


Figure 1.1: Penrose diagram of Schwarzschild space time. $i^{+(-)}$ is future (past) time-like infinity and i^0 is spatial infinity.

1.3.1 Cosmological Constant

The Einstein Hilbert action, with the addition of a cosmological constant, takes the following form [6][6]:

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2\Lambda] \tag{1.18}$$

If one varies the action with respect to the metric it results in the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{1.19}$$

Contracting both sides with $g^{\mu\nu}$ we get $R - \frac{1}{2}(4)R + \Lambda(4) = 0$ (since $g^{\mu\nu}g_{\mu\nu} = 4$) which implies that $R = 4\Lambda$. This makes the above EoM 1.19 become:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.20}$$

One can then proceed in the same manner as the Schwarzschild case (see chapter 5 of [6]), i.e. by making a spherically symmetric ansatz of the metric, calculating the Ricci tensor and, using 1.20 to derive the form of the functions in the ansatz. For a positive cosmological constant as we have in de Sitter space, the result is the following metric[7]:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2}dr^2 + r^2 d\Omega_2 \tag{1.21}$$

[6]G is here for aesthetic reasons we will subsequently set it to 1

[7]Note that the solution we present here is only the particular solution to $R_{\mu\nu} = \Lambda g_{\mu\nu}$. We could still have a $2M/r$ term which is a solution to the homogeneous solution $R_{\mu\nu} = 0$, but you may consider that we have set $M = 0$.

This bears resemblance to the Schwarzschild metric 1.7, just with a different $f(r) = 1 - \frac{\Lambda}{3}r^2$. This shows that there is a co-ordinate singularity at $f(r) = 0$, so $r = \frac{1}{\sqrt{\Lambda}}$. This corresponds to a *cosmological horizon* and can be physically interpreted as the radius after which information would need to propagate faster then the speed of light to overcome the affects of expansion.

1.3.2 Embedding Co-ordinates

n-dimensional de Sitter space is more often discussed in the context of it being embedded in an $n+1$ dimensional space. Typically de Sitter is described as a sub-manifold of $n+1$-dimension Minkowski space, which has the following metric:

$$ds^2 = -dX_0^2 + \sum_i^n dX_i^2$$

In this space $n-$dimensional de Sitter space is described as the hyperbolic surface (see Figure 1.2):

$$-X_0^2 + \sum_i X_i^2 = X_\mu X^\mu = \ell^2$$

Where ℓ has dimensions of length and is the characteristic scale of the space. We can achieve this metric and constraint for $n = 4$ from 4.1 with the following change of co-ordinates:

$$X_0 = \sqrt{\frac{3}{\Lambda} - r^2}\sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), \quad X_1 = \sqrt{\frac{3}{\Lambda} - r^2}\cosh\left(\sqrt{\frac{\Lambda}{3}}t\right), \quad X_i = ry_i$$

Where here y_i are co-ordinates on the $2-$sphere (i.e. $y_2 = \cos\phi\sin\theta,\ y_3 = \sin\phi\sin\theta,\ y_4 = \cos\theta$), such that $\sum_i dy_i^2 = d\Omega_2$. With this it can be verified that $X_\mu X^\mu = \frac{3}{\Lambda}$, so $\ell = \sqrt{\frac{3}{\Lambda}}$.

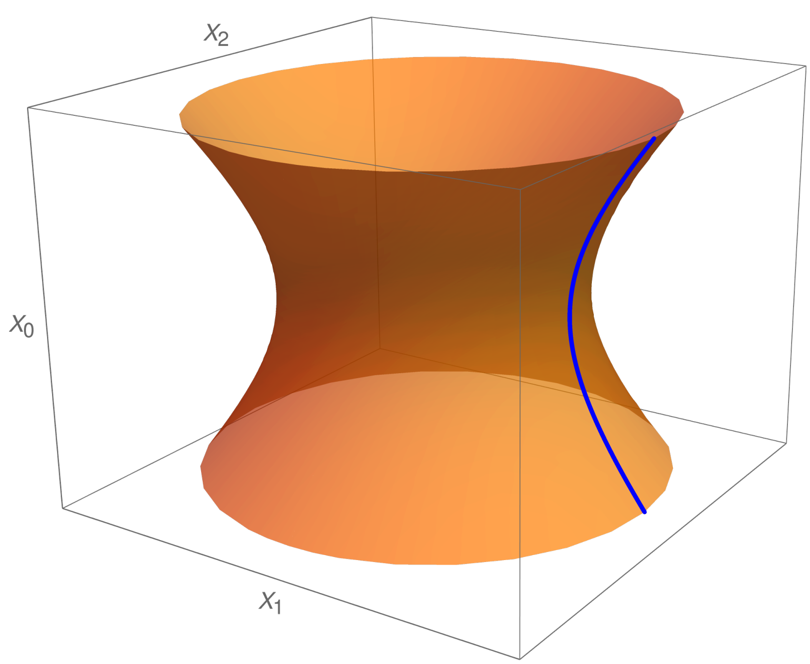
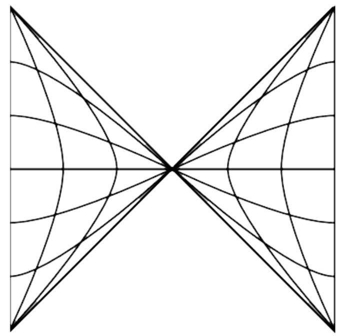


Figure 1.2: Example of deSitter embedding for 1 + 1-d deSitter space. The orange surface is the 2-d deSitter space and the blue line depicts the path of a observer on a geodesic who sits stationary at a constant position.



Chapter 2

The Unruh Effect

2.1 Introduction

From Einstein's theory of Special Relativity we are used to the notion of physics being the same in all inertial reference frames. That is, that the fundamental laws of physics are the same for every observer moving at a constant velocity relative to one and other. What happens if we betray Einstein and compare an inertial observer with a non-inertial (accelerating) frame; what effect will we see? What laws of physics or fundamental concepts, we thought completely sound, cease to hold?

The results of this are actually even more drastic then one might expect. This chapter seeks to show that the entire concept of *particles* is an observer dependent quantity. In other words, if a stationary observer, relative to some fixed origin, is sitting in what they perceive to be an empty vacuum; completely devoid of particles. Then an accelerating observer passing by *will not* see this same vacuum. Instead as will be shown, they will see space to have a *Temperature* T proportional to their acceleration. This is the *Unruh Effect* As one might expect the constants of proportionality of this temperature make it so that one would have to have a large acceleration to see such a temperature, hence why we do not see this at low energies. There is still debate as to whether we have already seen experimental evidence of the Unruh effect.

2.2 Uniform Acceleration

Before deriving this effect, let us consider the setup for our thought experiment (see Figure 2.1). We start by considering flat $1 + 1$-dimensional Minkowski spacetime. That is one spatial dimension and one temporal. Upon this we study the fluctuations of a massless scalar field ϕ and will consider the quantization of this scalar field in terms of two co-ordinate systems. The first is the regular Minkowski co-ordinates (t, x) that describe a stationary observer. The second set describe constantly accelerating observers.

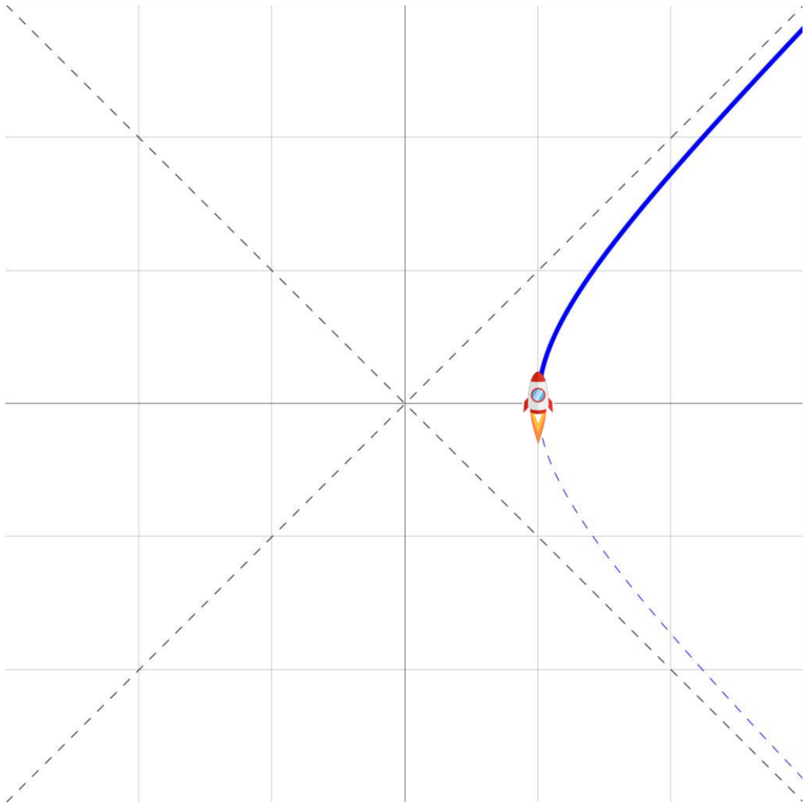


Figure 2.1: *Accelerating observer in a Minkowski diagram approaching the speed of light.*

As is shown in Appendix A.1, an observer with a constant acceleration a and proper time τ, moves in Minkowski space with:

$$t(\tau) = \frac{1}{a}\sinh(a\tau)$$
$$x(\tau) = \frac{1}{a}\cosh(a\tau)$$

We can generalize these to the so-called "Rindler co-ordinates" (η, ξ), which are related to (x, t) via:

$$t = \frac{1}{a}e^{a\xi}\sinh(a\eta) \tag{2.1}$$
$$x = \frac{1}{a}e^{a\xi}\cosh(a\eta) \tag{2.2}$$

In these co-ordinates constant accelerating observers (with acceleration α) have $\xi(\tau) = \frac{1}{a}\ln(\frac{a}{\alpha})$ and $\eta(\tau) = \frac{\alpha}{a}\tau$ (τ their proper time). So for $\alpha = a$, these reduce to the A.2 and A.3 and have $\xi = 0$ and $\eta = \tau$. These are basically the co-ordinates of an accelerated frame.

The metric in these co-ordinates is, from A.4:

$$ds^2 = e^{2\alpha\xi}\left[-d\eta^2 + d\xi^2\right] \tag{2.3}$$

Letting η and ξ range from $-\infty < \eta, \xi < \infty$, only covers a portion of the spacetime, as we can see from 2.2, since $\cosh\alpha\eta, e^{\alpha\xi} > 0 \implies x \geq 0$ for all ξ and η. This is shown in Figure 2.2 . This portion is called Region I. This region contains all accelerating observers with $a > 0$ and hyperbolic center at the origin. Any other observers accelerating the opposite direction $a < 0$ will be restricted to region II.

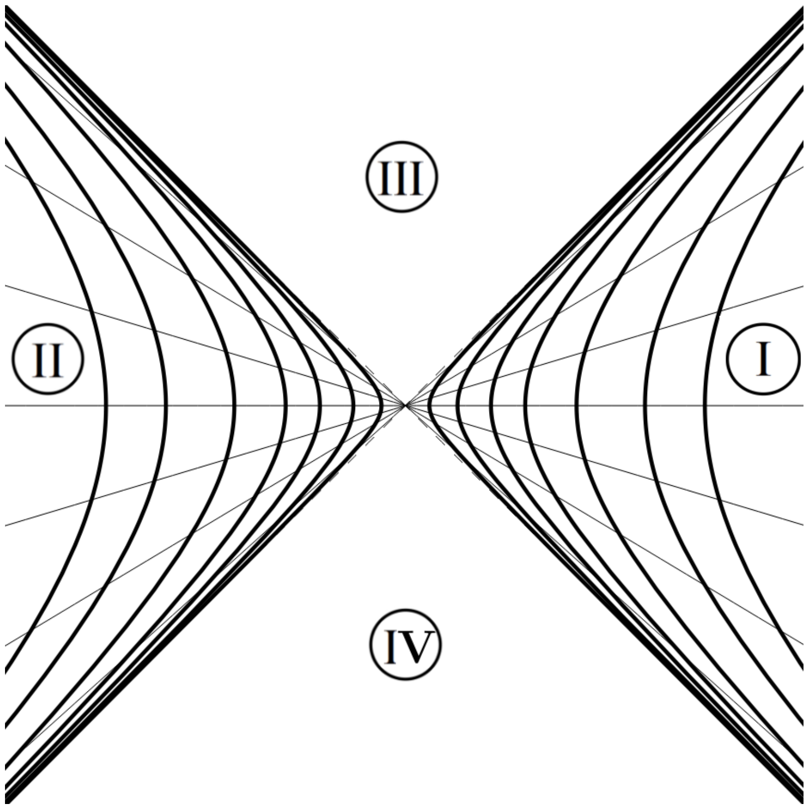


Figure 2.2: *Minkowski diagram showing Rindler co-ordinates, Hyperbolics are lines of constant* η *and straight lines are lines of constant* ξ. *These co-ordinates only cover Regions I and II*

This requires the introduction of a second set of co-ordinates to cover this other region of space:

$$t = -\frac{1}{a} e^{a\xi} \sinh(a\eta) \tag{2.4}$$

$$x = -\frac{1}{a} e^{a\xi} \cosh(a\eta) \tag{2.5}$$

These result in the same metric above 2.3. Note that while the sign in front of the t co-ordinate is not necessary to cover the right spacetime wedge , it is necessary to result in the same metric.

As we will see, the Unruh affect is dependent on having both of these sets of co-ordinates 2.1 , 2.2 and 2.4 , 2.5. It is important to cover all of Minkowski space, so that later when we can equate our two formulations of the same scalar field ϕ.

2.3 Expansion in Rindler Co-ordinates

The next to quantize the scalar field ϕ in the Rindler co-ordinates, in order to see what the field ϕ looks like from the perspective of an accelerating observer. It is expected that like in the flat space case, there will be some modes g_k that obey the Klein-Gordan equation and that ϕ can be expressed as a linear combination of these modes. Subsequently when the field is quantised, we find that these coefficients b_k must satisfy the commutation relations of the annihilation operators 1.6.

This requires the solving of the massless Klein-Gordon equation $\Box\phi = \partial_\mu\partial^\mu\phi = \partial_\mu g^{\mu\nu}\partial_\nu\phi = 0$. Using the form of $g^{\mu\nu}$ shown in A.4 we can see that $\Box\phi = 0$ then takes the form:

$$\Box\phi = e^{-2\alpha\xi}\left(-\partial_\eta^2 + \partial_\xi^2\right)\phi = 0$$
$$\implies \left(-\partial_\eta^2 + \partial_\xi^2\right)\phi = 0 \tag{2.6}$$

Therefore it is clear that in the Rindler co-ordinates the solutions to the KG equation are the same as they are in Minkowski co-ordinates. The solutions 1.3 can therefore be used, as they are just plane waves, so the modes take the form:

$$g_k = \frac{1}{\sqrt{4\pi\omega_k}} e^{\pm i\omega_k\eta \pm ik\xi}$$

Where for a massless scalar field $\omega_k = |k| > 0$.

However, recall that we need to be careful about what region of space time we are talking about as our co-ordinates transformations take two different forms in the two regions I and II. To account for this we define two sets of modes: $g_k^{(1)}$ (defined to be the modes in region I) and $g_k^{(2)}$, (defined to be the modes in Region II).

To have proper modes they must have positive frequency with respect to the future directed time-like killing vector. Appendix A.1.3 shows that the future directed time-like killing vector in Region I is ∂_η and in Region II is $\partial_{-\eta}$. However, a mode g_k is said to have "positive frequency" with respect to a given vector X if $Xg_k = -i\omega_k g_k$. This means the two sets of modes take the form:

$$g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega_k}} e^{-i\omega_k\eta + ik\xi} & \text{Region I} \\ 0 & \text{Region II} \end{cases} \tag{2.7}$$

$$g_k^{(2)} = \begin{cases} 0 & \text{Region I} \\ \frac{1}{\sqrt{4\pi\omega_k}} e^{+i\omega_k\eta + ik\xi} & \text{Region II} \end{cases} \tag{2.8}$$

Here each mode must be defined to be 0 in each others regions so that it remains okay to use the two different relations between (x, t) and (η, ξ). Otherwise a mode is not positive frequency with respect to a future directed time-like Killing vector. As can be verified, the signs in the exponential have been chosen such that:

$$\partial_\eta g_k^{(1)} = -i\omega_k g_k^{(1)}$$
$$\partial_{-\eta} g_k^{(2)} = -i\omega_k g_k^{(2)}$$

Meaning these are positive frequency modes across both Rindler regions.

With this it is possible to write down the entire scalar field expanded over these modes. The coefficients of these modes will be creation and annihilation operators. These are denoted $b_k^{(1,2)}$ as we have two sets of modes to cover both Rindler wedges. We also need to include the complex conjugates of these modes for completion.

$$\phi = \int_{-\infty}^{\infty} dk \left[b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*} \right] \tag{2.9}$$

These 4 modes cover not only the two Rindler wedges (Regions I & II) but can also be extended to cover the entire Minkowski space. This appears strange as these modes have been written explicitly in terms of η and ξ which in their full range only cover Regions I & II, but what we can do is *analytically* extend these modes with complex values if η and ξ to suitable cover all of the space. In appendix A.1.5 it is shown that these modes are appropriately normalized with respect to the KG inner product.

2.4 Extension to Minkowski modes

It has been shown how a scalar field ϕ can be expressed in terms of two sets of co-ordinates, Minkowski 1.3 and Rindler 2.9, that describe two different types of observers, stationary and accelerating. We have seen that in each case the modes that ϕ is expanded over are different and thus have different creation and annihilation operators. This motivates the question of how the Rindler vacuum might differ from the Minkowski one. Recall that a Rindler observer (in Region I) will observe a state $|0_R\rangle$ to be a vacuum or devoid of particles if $b_k^{(1)}|0_R\rangle = 0$. Similarly the same was done for the Minkowski vacuum, describing it as the state $|0_M\rangle$ where $a_k|0_M\rangle = 0$. The question we want to ask is then, if we instead act the annihilation operator in the Rindler expansion on the Minkowski vacuum $b_k^{(1)}|0_M\rangle$, what do we get? As we will see later the result is non-zero meaning that from the perspective of a Rindler observer the Minkowski vacuum is not empty.

To calculate $b_k^{(1)}|0_M\rangle$ one approach would be to try to find the appropriate form of $|0_M\rangle$ and seeing what the operation of $b_k^{(1)}$ does to it. A faster way would be find some way of expressing $b_k^{(1)}$ in terms of the Minkowski creation and annihilation operators a_k and $a_k^\dagger$, after which the action of $b_k^{(1)}$ on $|0_M\rangle$ would follow. This is the approach we will now take.

Starting with the relations in Region I, 2.1 and 2.2, by linear combinations exponentials of η and ξ can be constructed:

$$a(x-t) = e^{a\xi}e^{-a\eta} = e^{a(\xi-\eta)} \tag{2.10}$$

$$a(x+t) = e^{a\xi}e^{a\eta} = e^{a(\xi+\eta)} \tag{2.11}$$

This can also be done in Region II using relations 2.4 and 2.5:

$$a(-x+t) = e^{a\xi}e^{-a\eta} = e^{a(\xi-\eta)} \tag{2.12}$$

$$a(-x-t) = e^{a\xi}e^{a\eta} = e^{a(\xi+\eta)} \tag{2.13}$$

From these relations the Rindler modes can be expressed in terms of x and t. But first our modes must be split into Right and Left moving modes. This can be achieved by writing the modes in light-cone co-ordinates, $\tilde{u} = \eta - \xi$ and $\tilde{v} = \eta + \xi$. This essentially separates out the modes into two groups classified by $k > 0$ and $k < 0$. This then produces $g_k^{(1)} = g_L^{(1)}(k) + g_R^{(1)}(k)$, where:

$$g_R^{(1)}(k) = \Theta(k)g_k^{(1)} \tag{2.14}$$

$$g_L^{(1)}(k) = \Theta(-k)g_k^{(1)} \tag{2.15}$$

This way the integral expansion 2.9 takes the form:

$$\phi = \int_0^\infty dk \left[b_k^{(1)} g_R^{(1)}(k) + b_k^{(1)\dagger} g_R^{(1)*}(k) + b_k^{(2)} g_R^{(2)}(k) + b_k^{(2)\dagger} g_R^{(2)*}(k) \right]$$
$$+ \int_{-\infty}^0 dk \left[b_k^{(1)} g_L^{(1)}(k) + b_k^{(1)\dagger} g_L^{(1)*}(k) + b_k^{(2)} g_L^{(2)}(k) + b_k^{(2)\dagger} g_L^{(2)*}(k) \right] \quad (2.16)$$

Now by examining form of these modes it is possible to see if they can be related to 2.10 and 2.12. We will start by examining the right moving modes with $k > 0$ but it will be clear that the discussion will be similar for left moving modes. Recall that $\omega_k = |k|$, so for $k > 0$, $\omega_k = k \equiv \omega$, so using 2.7 and 2.10:

$$\sqrt{4\pi\omega} g_R^{(1)}(k) = e^{i\omega(\xi-\eta)} = (ax - at)^{\frac{i\omega}{a}}$$
$$= a^{\frac{i\omega}{a}} (x - t)^{\frac{i\omega}{a}} \quad (2.17)$$

This is good, the mode is expressed purely in x and t, but it does not cover all of Minkowski space. To achieve this modes from Region II must be included. Let us see what $g_R^{(2)}(k)$ looks like:

$$\sqrt{4\pi\omega} g_R^{(2)}(k) = e^{i\omega(\xi+\eta)} = (-ax - at)^{\frac{i\omega}{a}}$$
$$= a^{\frac{i\omega}{a}} (-x - t)^{\frac{i\omega}{a}} \quad (2.18)$$

This is not quite the same as what we had for $g_R^{(1)}(k)$, so they cannot be combined. The next most obvious step would be to try use $g_R^{(2)*}(k)$, but this changes both of the signs of x and t, where as we only want to change the sign of x. Instead by taking the complex conjugate and reversing the sign of k we indeed get what we need. However, flipping the sign of k means using a negative value, meaning it is necessary to use one of the *left moving* modes. Since these are only defined for $k < 0$, they must be evaluated at $g_L^{(2)*}(-k)$. To show this works, the following can be written using $\omega = k > 0$:

$$\sqrt{4\pi\omega} g_L^{(2)*}(-k) = e^{-i\omega(-\xi+\eta)} = (-ax + at)^{\frac{i\omega}{a}}$$
$$= (-1)^{\frac{i\omega}{a}} a^{\frac{i\omega}{a}} (x - t)^{\frac{i\omega}{a}} \quad (2.19)$$

In appendix A.1.6 it is shown that the two modes $g_R^{(1)}(k)$ and $g_L^{(2)*}(-k)$ do not overlap and cover the entire spacetime as intended [1], this means we can write them down as a combination:

$$\sqrt{4\pi\omega} \left(g_R^{(1)}(k) + (-1)^{-\frac{i\omega}{a}} g_L^{(2)*}(-k) \right) = a^{\frac{i\omega}{a}} (x - t)^{\frac{i\omega}{a}} \quad (2.20)$$

From this expression here the "analytic extension" discussed earlier is clear, this expression is simply used for all values of t and x in Regions III and IV. It is not necessary to say what values of ξ and η they correspond to.

Similarly for Region II the same procedure can be carried out to get modes of the same form as 2.18:

$$\sqrt{4\pi\omega} \left(g_R^{(2)}(k) + (-1)^{-\frac{i\omega}{a}} g_L^{(1)*}(-k) \right) = a^{\frac{i\omega}{a}} (-x - t)^{\frac{i\omega}{a}} \quad (2.21)$$

[1] Note that we have not stated the value of $(-1)^{\frac{i\omega}{a}}$ as there are multiple choice, we choose one of these later in section 2.5.1

These are also non-overlapping and cover all of spacetime. This discussion has been for right moving modes, the exact same procedure can be done for left moving modes. Here $\omega_k = |k|$, so for $k < 0$; $\omega_k = -k \equiv \omega$:

$$\begin{aligned}\sqrt{4\pi\omega} g_L^{(1)}(k) &= e^{i\omega(-\xi-\eta)} = (ax+at)^{-\frac{i\omega}{a}} \\ &= (-1)^{-\frac{i\omega}{a}} a^{-\frac{i\omega}{a}} (-x-t)^{-\frac{i\omega}{a}}\end{aligned}$$

Taking inspiration from the construction of $g_k^{(1)*}$ a relation of the same form can be constructed:

$$\begin{aligned}\sqrt{4\pi\omega} g_R^{(2)*}(-k) &= e^{-i\omega(\xi+\eta)} = (-ax-at)^{-\frac{i\omega}{a}} \\ &= a^{-\frac{i\omega}{a}} (-x-t)^{-\frac{i\omega}{a}}\end{aligned}$$

The same idea holds for $g_L^{(2)}$ so the two combinations can be written as:

$$\sqrt{4\pi\omega}\left((-1)^{\frac{i\omega}{a}} g_L^{(1)}(k) + g_R^{(2)*}(-k)\right) = a^{-\frac{i\omega}{a}}(-x-t)^{-\frac{i\omega}{a}} \tag{2.22}$$

and

$$\sqrt{4\pi\omega}\left((-1)^{\frac{i\omega}{a}} g_L^{(2)}(k) + g_R^{(1)*}(-k)\right) = a^{-\frac{i\omega}{a}}(x-t)^{-\frac{i\omega}{a}} \tag{2.23}$$

We can then recognize these as complex conjugates of 2.21 and 2.20 respectively. This means 2.20 and 2.21 are modes that cover all of Minkowski space for all values of k, so we can combine 2.20 and the conjugate of 2.23 to write:

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + (-1)^{-\frac{i\omega}{a}} g_{-k}^{(2)*}\right) = a^{\frac{i\omega}{a}}(x-t)^{\frac{i\omega}{a}} \tag{2.24}$$

And similarly, 2.21 and the conjugate of 2.22 to write:

$$\sqrt{4\pi\omega}\left(g_k^{(2)} + (-1)^{-\frac{i\omega}{a}} g_{-k}^{(1)*}\right) = a^{\frac{i\omega}{a}}(-x-t)^{\frac{i\omega}{a}} \tag{2.25}$$

The combinations 2.24 and 2.25, along with their complex conjugates always need to be included, then give enough to expand our scalar field ϕ over all of space time. This expansion takes the form:

$$\phi = \int_{-\infty}^{\infty} dk \left[c_k^{(1)} h_k^{(1)} + c_k^{(1)\dagger} h_k^{(1)*} + c_k^{(2)} h_k^{(2)} + c_k^{(2)\dagger} h_k^{(2)*}\right] \tag{2.26}$$

Where the modes $h_k^{(1,2)}$ are defined as:

$$h_k^{(1)} = \frac{1}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}\left[g_k^{(1)} + (-1)^{\frac{i\omega}{a}} g_{-k}^{(2)*}\right] \tag{2.27}$$

$$h_k^{(2)} = \frac{1}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}\left[g_k^{(2)} + (-1)^{\frac{i\omega}{a}} g_{-k}^{(1)*}\right] \tag{2.28}$$

Where the factor of $A(\omega) = \frac{1}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}$ is for normalisation, which is derived in Appendix A.1.7.

2.5 "Good" Minkowski Modes

The modes that have just been constructed are orthonormal and cover all of spacetime. But how do we know that these are appropriate Minkowski modes?. "Good" modes are ones which are positive frequency with respect to a future directed time-like killing vector. This, for example, rules out our Rindler modes 2.7 and 2.8 as we is showed in Appendix A.1.3 the future directed time-like killing vector is different in the two Regions I and II. This means there is no way for a single Rindler mode to be written in terms of positive frequency Minkowski modes. What will now be shown is that the combinations 2.20 and 2.21 are in fact made of combinations of positive frequency Minkowski modes.

As a reminder let us consider the Minkowski modes. In 1.3 we had that the scalar field can be expanded over the modes[2]:

$$\begin{aligned}\phi &= \int_{-\infty}^{\infty} dp \left[\hat{a}_p e^{-i\lambda_p t+ipx} + \hat{a}_p^\dagger e^{i\lambda_p t-ipx}\right] \qquad (\lambda_p = |p|)\\ &= \int_{0}^{\infty} dp \left[\hat{a}_p e^{-ip(t-x)} + \hat{a}_p^\dagger e^{ip(t-x)}\right] + \int_{-\infty}^{0} dp \left[\hat{a}_p e^{ip(t+x)} + \hat{a}_p^\dagger e^{-ip(t+x)}\right]\end{aligned}$$

The integration variable in the second integral is then changed from $p \to -p$, however, since this is for $p < 0$, which means $\lambda = |p| = -p$, then this is just changing the integration variable to λ, while in the first integral $\lambda = |p| = p$. These two integrals can be combined to give:

$$\begin{aligned}&= \int_{0}^{\infty} d\lambda \left[\hat{a}_p e^{-i\lambda(t-x)} + \hat{a}_p^\dagger e^{i\lambda(t-x)} + \hat{a}_p e^{-i\lambda(t+x)} + \hat{a}_p^\dagger e^{i\lambda(t+x)}\right]\\ &= \int_{0}^{\infty} d\lambda \left[\hat{a}_p e^{-i\lambda u} + \hat{a}_p^\dagger e^{i\lambda u} + \hat{a}_p e^{-i\lambda v} + \hat{a}_p^\dagger e^{i\lambda v}\right] \qquad (2.29)\end{aligned}$$

These are the right and left moving modes in Minkowski co-ordinates.

Now we need to analyze the RHS of 2.20 and 2.21 as complex functions. This is made clear by writing these modes in terms of the light-cone co-ordinates[3] $v = t + x$ and $u = t - x$. Then by considering the Fourier transforms of these functions:

$$a^{\frac{i\omega}{a}}(x-t)^{\frac{i\omega}{a}} = a^{\frac{i\omega}{a}}(-u)^{\frac{i\omega}{a}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(\lambda) e^{-i\lambda u} d\lambda \qquad (2.30)$$

$$a^{\frac{i\omega}{a}}(-x-t)^{\frac{i\omega}{a}} = a^{\frac{i\omega}{a}}(-v)^{\frac{i\omega}{a}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(\lambda) e^{-i\lambda v} d\lambda \qquad (2.31)$$

We can notice that if it happened to be the case that for $\lambda < 0$, $\tilde{h}(\lambda) = 0$. Subsequently these modes would in fact be sums of positive frequency Minkowski modes. Using some complex analysis it will now be shown that this is indeed the case for $(x-t)^{\frac{i\omega}{a}}$ and $(-x-t)^{\frac{i\omega}{a}}$.

[2]Here we use the notation of p being momentum and $\lambda_p = |p|$ the frequency for the Minkowski co-ordinates to distinguish from the Rindler momentum k and frequency ω_k.

[3]We can see from this that the ability of these modes to be written in terms of light-cone co-ordinates is the key factor that lets us construct proper Minkowski modes

2.5.1 Complex Analysis Trick

We begin by stating the following theorem (See theorem 19.2 of [16]):

> *A function is a combination of only positive frequency modes iff as a complex function it is analytic and bounded in the lower half complex plane.*

Why this theorem is true is discussed in appendix A.1.8. The theorem provides that if as a complex functions 2.20 and 2.21 are analytic and bounded in the lower half complex plane then they must be combinations of only positive frequencies, i.e. $\tilde{h}(\lambda) = 0,\ \forall\ \lambda < 0$. This is all that is needed to prove these are "good" modes. This begins by writing the 2.20 in the following way. Following [9] we notice that 2.21 is essentially the same function with $u \leftrightarrow v$:

$$a^{\frac{i\omega}{a}}(-u)^{\frac{i\omega}{a}} = a^{\frac{i\omega}{a}} e^{i\frac{\omega}{a}\ln(-u)}$$

Now since this considers u as a complex number, there is the issue that the function $\ln(-u)$ is multi-valued. To fix a branch cut is applied and hence the way in which we rotate around the origin is picked. Since we are interested in just the real parts of u and the lower half plane it suffices to say that we want to pick the branch cut in the upper half plane. We can expand $\ln(-u)$ in polar co-ordinates as follows:

$$\ln(-u) = \ln|-u| + i\theta + i(2k-1)\pi \tag{2.32}$$

Here, $\ln|-u|$ is the single valued natural log that acts on real numbers, as the magnitude of u is real. θ is the angle of u in the complex plane and it is the $i(2k+1)\pi$ that gives this function its multi-valued nature. Note we are used to seeing this as $2k\pi$, i.e. an addition of an even multiple of π since $k \in \mathbb{Z}$. However, the analytic function chosen is expected to have $\mathrm{Ln}(1) = 2n\pi$ $(n \in \mathbb{Z})$, which happens when $u = -1$, since in this case it is the lower half complex plane we are interested in. This corresponds to $|u| = |-1| = 1$ and $\theta = -\pi$[4], in polar co-ordinates. This results in $\ln(1) = \ln|1| - i\pi + i(2k-1)\pi = ((2k-1)-1)\pi = 2(k-1)\pi = 2n\pi$ (setting $n = k-1$), which would not be the case if an even multiple was chosen for addition.

With this established the simplest case $k = 0$ is then chosen, so that our single valued function is:

$$\mathrm{Ln}(-u) = \ln|-u| + i(\theta - \pi), \quad \theta \in (-\frac{3\pi}{2}, \frac{\pi}{2}) \tag{2.33}$$

With this choice of restriction on θ it is clear that the branch cut is along the imaginary axis in the upper half complex plane as needed. With this choice it is possible to check that:

$$a^{\frac{i\omega}{a}} e^{i\frac{\omega}{a}\ln(-u)} = a^{\frac{i\omega}{a}} e^{i\frac{\omega}{a}[\ln|-u| + i(\theta-\pi)]}$$

Is now clearly bounded in the lower half plane as $e^{i\frac{\omega}{a}\ln|-u|}$ is purely oscillatory and $e^{i\frac{\omega}{a}i(\theta-\pi)}$ is bounded as θ is bounded. Finally a version of $\ln(-u)$ has been constructed that is analytic. As well as this the exponential function is analytic, and the composition of analytic functions is itself an analytic function. Hence, we have function that is analytic and bounded in the lower half

[4] This means rotation is clockwise through the complex plane, $\theta = \pi$ would be a anti-clockwise rotation in the upper half plane.

complex plane. This is all we needed to confirm that the $h_k^{(1,2)}$ are "Good" Minkowski modes as they are functions of positive frequency modes only.

As a final note, it is clear that the choice made to get a single valued log in 2.33 helps avoid an ambiguity. In 2.20 and 2.21 we have the factor $(-1)^{-\frac{i\omega}{a}}$ which can now be calculated using 2.33:

$$\begin{aligned}(-1)^{-\frac{i\omega}{a}} &= e^{-i\frac{\omega}{a}\mathrm{Ln}(-1)} = e^{-i\frac{\omega}{a}[\ln|-1|+i(0-\pi)]} \quad (u,v=1 \implies \theta=0)\\ &= e^{-\frac{\pi\omega}{a}}\end{aligned} \tag{2.34}$$

2.6 Unruh Temperature

At this point almost everything needed to perform the calculation is accounted for. Recall as discussed in 2.4, what we are looking for is a relation between the the Rindler $b_k^{(1)}$ and Minkowski c_k creation and annihilation operators. This can be achieved by expanding the same scalar field ϕ over the Rindler modes as in 2.9 and the newly constructed Minkowski modes in 2.26:

$$\begin{aligned}\phi &= \int_{-\infty}^{\infty} dk\left[b_k^{(1)}g_k^{(1)} + b_k^{(1)\dagger}g_k^{(1)*} + b_k^{(2)}g_k^{(2)} + b_k^{(2)\dagger}g_k^{(2)*}\right]\\ &= \int_{-\infty}^{\infty} dk\left[c_k^{(1)}h_k^{(1)} + c_k^{(1)\dagger}h_k^{(1)*} + c_k^{(2)}h_k^{(2)} + c_k^{(2)\dagger}h_k^{(2)*}\right]\\ = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}&\left[c_k^{(1)}\left(g_k^{(1)} + e^{-\frac{\pi\omega}{a}}g_{-k}^{(2)*}\right) + c_k^{(1)\dagger}\left(g_k^{(1)*} + e^{-\frac{\pi\omega}{a}}g_k^{(2)}\right)\right.\\ &\left. + c_k^{(2)}\left(g_k^{(2)} + e^{-\frac{\pi\omega}{a}}g_{-k}^{(1)*}\right) + c_k^{(2)\dagger}\left(g_k^{(2)*} + e^{-\frac{\pi\omega}{a}}g_{-k}^{(1)}\right)\right]\end{aligned}$$

The KG inner product A.7 can be used to isolate the creation and annihilation operators. From the first expansion of ϕ it is clear that $(g_{k'}^{(1)}, \phi) = b_{k'}^{(1)}$, but if this calculated using the second expansion we see that:

$$\begin{aligned}b_{k'}^{(1)} = (g_{k'}^{(1)}, \phi) &= \frac{1}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}\left(c_k^{(1)}\delta(k-k') + e^{-\frac{\pi\omega}{a}}c_k^{(2)\dagger}\delta(k+k')\right)\\ &= \frac{1}{\sqrt{1-e^{-\frac{2\pi\omega}{a}}}}\left(c_{k'}^{(1)} + e^{-\frac{\pi\omega}{a}}c_{-k'}^{(2)\dagger}\right)\end{aligned} \tag{2.35}$$

Where relations A.8 and A.9 have been used. Similar identifications can be made for the other operators.

Now we are able to perform the calculation. Returning to our opening thought experiment we have a Rindler observer accelerating in Region I of the space time and we want to know what this observer would see if they accelerated through a Minkowski vacuum $|0_M\rangle$. What they "see" can be quantified by the expectation value of the particle number operator, which in this Region is just $\langle N_k\rangle = \langle 0_M| b_k^{(1)\dagger}b_k^{(1)}|0_M\rangle$. Then using our above calculated 2.35 and remembering that since the $h_k^{(1,2)}$ modes are perfectly valid Minkowski modes, then the creation and annihilation operators $c_k^{(1)}$ and $c_k^{(1)\dagger}$ interact regularly with the Minkowski vacuum $|0_M\rangle$, in that $c_k^{(1)}|0_M\rangle = \langle 0_M| c_k^{(1)\dagger} = 0$.

This means:

$$\begin{aligned}\langle N_k\rangle &= \langle 0_M| b_k^{(1)\dagger} b_k^{(1)} |0_M\rangle \\ &= \frac{1}{1-e^{-\frac{2\pi\omega}{a}}} \langle 0_M| \left(c_k^{(1)\dagger} + e^{-\frac{\pi\omega}{a}} c_{-k}^{(2)}\right)\left(c_k^{(1)} + e^{-\frac{\pi\omega}{a}} c_{-k}^{(2)\dagger}\right) |0_M\rangle \\ &= \frac{e^{-\frac{2\pi\omega}{a}}}{1-e^{-\frac{2\pi\omega}{a}}} \langle 0_M| c_{-k}^{(2)} c_{-k}^{(2)\dagger} |0_M\rangle\end{aligned}$$

Then seeing as $c_k^{(1)}$ and $c_k^{(1)\dagger}$ satisfy the commutation relations 1.6, we must have that, $\langle 0_M| c_k^{(2)} c_k^{(2)\dagger} |0_M\rangle = \langle 0_M| [c_k^{(2)}, c_k^{(2)\dagger}] + c_k^{(2)\dagger} c_k^{(2)} |0_M\rangle = \delta(0)\langle 0_M| |0_M\rangle = \delta(0)$:

$$\langle N_k\rangle = \frac{\delta(0)}{e^{\frac{2\pi\omega}{a}}-1} \tag{2.36}$$

This divergent factor is due to the fact that we are considering the volume of an unbounded space. If instead this calculation was repeated in a finite box, the momenta k would be discrete and the factor $\delta(0)$ would be equal to the volume V. This can be seen from the fact that the delta function arises from $\delta(k-k)$, where k is the momentum. If this is expanded through the integral definition of the delta function, which must be an integral over space as that is the conjugate of momentum:

$$\delta(0) = \int_V e^{ix(k-k)} dx = \int_V dx = V$$

Where we are still in one spatial dimension so the volume is a length, but the result holds in higher spatial dimensions. What we should really concern ourselves with is the *number density* of particles defined as:

$$\boxed{\langle n_k\rangle = \frac{\langle N_k\rangle}{V} = \frac{1}{e^{\frac{2\pi\omega}{a}}-1}} \tag{2.37}$$

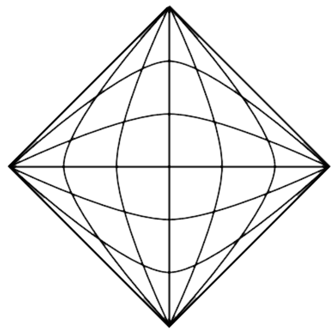
This is the occupancy number for a Planck distribution with temperature:

$$\boxed{T = \frac{a}{2\pi}} \tag{2.38}$$

So an accelerating observer will see the Minkowski vacuum as a thermal bath with a temperature! Returning to SI units and restoring the factors we get that this temperature is:

$$T = \frac{\hbar a}{2\pi c k} \simeq 4.055 \times 10^{-21} a \text{ [K]}$$

For any reasonable value of acceleration (must be in m/s^2) this temperature is *tiny*, hence why it has been so hard to experimentally detect.



Chapter 3

Hawking Radiation

3.1 Introduction

The Unruh affect, while in flat spacetime teaches an important lesson about QFT in curved spacetime, that being that the idea of the "vacuum" and "particles" are observer dependent quantities rather than fundamental concepts [6]. Hawking Radiation as will be shown is a similar phenomenum. Here, two different notions of a vacuum arise, due to their natural co-ordinates being different. This time however, the difference in co-ordinate systems arises, not just because one observer is accelerating, but due to the *geometry of spacetime* as the presence of a black hole gives rise to an event horizon.

The approach taken for the Unruh affect made use of the ease of calculation in directly constructing Minkowski modes and the ability to show that they are "good" positive frequency modes that cover all of spacetime. In the case of the Hawking radiation, this is not so easy. So a more general approach will have to be taken that will rely on some tricks to get around direct calculations.

The scenario we will be considering is the same as Hawking in his original paper [13]. This scenario consists of a massless scalar field in the classical spacetime background of a spherically symmetric dust cloud that collapses to form a Schwarzschild black hole. It will be found that if no particles were present at past infinity, (i.e. there was a vacuum), then a distant observer at future infinity will observe a spectrum of particles, owing to the fact that the spacetime close to the black hole, where the particles appear to come, has a different notion of a vacuum to past infinity.

3.2 Wave equation

In flat space-time the wave equation $\Box\varphi = 0$ was utilized, for our massless scalar field[1]. In order to generalize this to curved space time we need to look at the Lagrangian. The scalar Klein Gordon field was discussed in 1.1.1. These KG fields satisfy the KG equation 1.1, which has the following

[1]Note in this section we use φ to refer to the scalar field to avoid confusion with the azimuthal angle ϕ.

corresponding Lagrange density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \tag{3.1}$$

It can then be recalled that in order for a theory to be transported to curved space time, two things need to happen. First we need to make sure that our Lagrangian is a scalar. That is, it is invariant under co-ordinate transformations, that may have different determinants of the metric g. We showed in section 2 that in order for the volume element to transform like a tensor we need to multiply it by $\sqrt{|g|}$. This means in the action we have:

$$S = \int \mathcal{L}d^nx \to \int \sqrt{|g|}\mathcal{L}d^nx$$

When we translate to curved spacetime. Note that in the Minkowski case $\sqrt{|g|} = 1$ and the Lagrange density is the same as above 3.1. The second thing we need to do in curved spacetime is to take our laws of physics in flat spacetime and replace the partial derivatives with co-variant derivatives. This makes the equations tensor equations and means they account for any curvature of space. This means our Lagrange density becomes[2]:

$$\mathcal{L} = \frac{1}{2}\sqrt{|g|}\left(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - m^2\varphi^2\right) \tag{3.2}$$

We show in appendix B.1.5 that in curved space-time the equations of motion for the action with a Lagrange density $\mathcal{L}$, varied with respect to a scalar field φ are:

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\varphi\right)}\right) = 0$$

We can easily then compute from 3.2 that $\frac{\partial\mathcal{L}}{\partial\varphi} = -\sqrt{|g|}m^2\varphi$ and $\frac{\partial\mathcal{L}}{\partial(\nabla_\sigma\varphi)} = \sqrt{|g|}g^{\mu\nu}\nabla_\mu\varphi\delta^\sigma_\nu = \sqrt{|g|}g^{\mu\sigma}\nabla_\mu\varphi$. Which mean the EoM are:

$$\frac{1}{\sqrt{|g|}}\partial_\sigma\left(\sqrt{|g|}g^{\mu\sigma}\nabla_\mu\varphi\right) - m^2\varphi = 0 \tag{3.3}$$

It can be shown (see appendix B.1.6) that this first term is equal to:

$$\frac{1}{\sqrt{|g|}}\partial_\sigma\left(\sqrt{|g|}g^{\mu\sigma}\nabla_\mu\varphi\right) = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi \equiv \Box\varphi$$

So we can see with this extension of the definition of the $\Box$ operator to $g^{\mu\nu}\nabla_\mu\nabla_\nu$ [3] we get the same EoM for a massless scalar field that we had in flat space-time $\Box\varphi = 0$.

[2]Note that we are choosing the case of minimal coupling, meaning there is no term linking φ to the Ricci scalar R, so no back reaction of the field on the metric.

[3]You can see this is indeed an extension, as in the absence of curved spacetime this reduced to partial derivatives as normal.

3.2.1 Schwarzschild Wave Equation

We can now plug in the Schwarzschild solution 1.7 into $\frac{1}{\sqrt{|g|}}\partial_\sigma\left(\sqrt{|g|}g^{\mu\sigma}\nabla_\mu\phi\right) = 0$. Using the fact that the determinant of the metric is $g = -r^4\sin^2\theta$, term by term this is:

$$\partial_t\left(-\frac{\partial_t\varphi}{\left(1-\frac{2M}{r}\right)}\right) + \frac{1}{r^2}\partial_r\left(r^2\left(1-\frac{2M}{r}\right)\partial_r\varphi\right) + \frac{1}{\sin\theta}\partial_\theta\left(\frac{\sin\theta}{r^2}\partial_\theta\varphi\right) + \partial_\phi\left(\frac{1}{r^2\sin^2\theta}\partial_\phi\varphi\right) = 0$$

We can recognize that the last two terms give us the familiar angular equation for the spherical harmonics. This motivates us to make the ansatz:

$$\varphi = \frac{1}{r}f(r,t)Y_{\ell m}(\theta,\phi) \tag{3.4}$$

Where we have added the factor of $1/r$ is added to make things simpler. We can then plug this in to our wave equation:

$$-\frac{Y_{\ell m}}{r\left(1-\frac{2M}{r}\right)}\partial_t^2 f + \frac{Y_{\ell m}}{r^2}\partial_r\left(r^2\left(1-\frac{2M}{r}\right)\partial_r\left[\frac{f}{r}\right]\right) + \frac{f}{r^3}\nabla^2_{\theta,\phi}Y_{\ell m} = 0 \tag{3.5}$$

Where $\nabla^2_{\theta,\phi} = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial^2_\phi$. The spherical harmonics satisfy:

$$\nabla^2_{\theta,\phi}Y_{\ell m} = -\ell(\ell+1)Y_{\ell m} \tag{3.6}$$

Where ℓ is the angular quantum number, ℓ must be a non-negative integer and $-\ell \leq m \leq \ell$. In appendix B.2.1 we show that when we use the tortoise co-ordinate r^* 1.10, 3.5 reduces to:

$$\begin{aligned} &-\partial_t^2 f + \partial_{r^*}^2 f - \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)f = 0 \\ &\left[-\partial_t^2 + \partial_{r^*}^2 - V(r)\right]f = 0 \end{aligned} \tag{3.7}$$

We will see when we go too look at the radiation from the black hole the two ranges of r we will be interested in, will be $r >> 2M$ and $r \sim 2M$. In both cases we can see that, $V(r >> 2M) = 0$ as the $\frac{1}{r^3}$ and $\frac{1}{r^2}$ vanish, and $V(r \sim 2M) = 0$ as here $\left(1-\frac{2M}{r}\right) \approx 0$. This means in the two regimes of interest, when using the tortoise co-ordinate r^*, the solutions to the the wave equation, for $f(t, r^*)$, are just plane waves.

3.3 Hawking's Calculation

We will now proceed to outline Hawking's original calculation [15]. The setup we will be describing is shown in a Penrose diagram in Figure 3.1. As outlined before what we will be measuring is the spectrum of particles that appear at future null infinity, which is denoted $\mathscr{I}^+$ in Figure 3.1. In the conformal Penrose diagram light rays still travel at 45° [4]. Hawking used this fact to trace a light ray *backwards in time*, starting at some point on future null infinity $\mathscr{I}^+$ back to $r = 0$, where the light ray bounced of the center of the star and traveled all the back to some point past null infinity $\mathscr{I}^-$. This path is denoted by the co-ordinate v as can be seen in 3.1. The latest possible path is denoted by v_0, this path traces out the event horizon at $r = 2M$ before propagating to future time-like infinity.

[4] By "light rays travel at 45°" we mean lines of constant phase are always at 45°. In appendix A.1.6 we outlined that lines of constant phase represent the direction the waves are traveling in

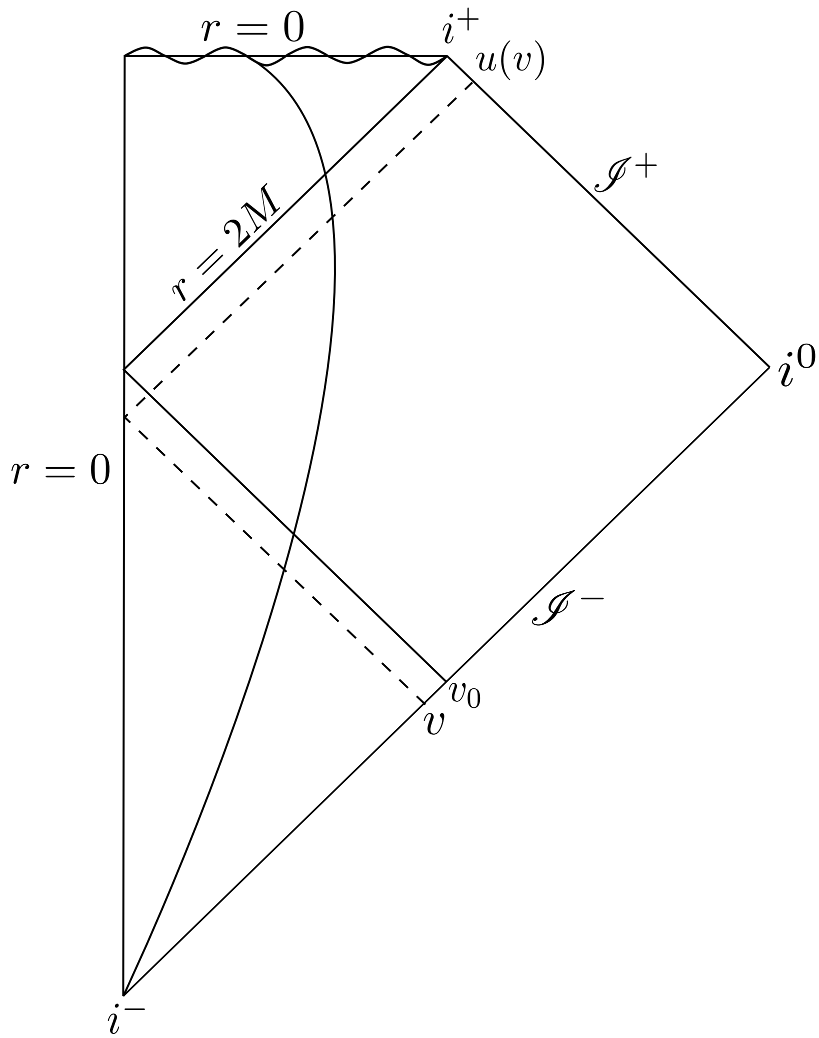


Figure 3.1: Penrose diagram of the collapsing star. Curved line shows the exterior of the collapsing star. Line labeled v_0 is the latest possible light ray that could escape the collapsing body, this is at $r = 2M$, v is the general light ray that escapes. $i^{+(-)}$ is future (past) time-like infinity, i^0 is spatial infinity and $\mathscr{I}^{+(-)}$ is future (past) null infinity. [6]

3.3.1 Field Expansion In The Past

We now want to figure out how the fields can be expanded in the two limits we are interested in. The first limit is far outside the collapsing body in the past (close to $\mathscr{I}^-$). This is where we expect the incoming ray to come from. Here, the scalar field φ that we will be talking about is only comprised of *ingoing solutions*. This is because these modes are the only modes that can reach the black hole and become relevant for the thermal spectrum from the black hole. The outgoing modes on $\mathscr{I}^-$ never near the black hole. As outlined in 1.2.7 these ingoing modes are characterized by having constant $v = t + r^*$. We can then write down the positive frequency modes of energy ω, which we know from the ansatz 3.4 and the wave equation 3.7, will asymptotically take the form:

$$f_\omega \propto \frac{1}{r\sqrt{\omega}} e^{-i\omega v} Y_{lm}$$

Where we have added the $1/\sqrt{\omega}$ as we expect the normalization via the scalar product to involve such a factor, but we have neglected the overall normalization factor as it is constant and unnecessary for or following procedure. We can then expand the the scalar field φ over these modes, for all frequencies. Note that we can use frequency here as our integration variable instead of the momenta k as we are only dealing with inward moving modes, which have $k < 0$, so there is no ambiguity about $\omega = \pm|k|$. This means φ takes the form:

$$\varphi = \int_0^\infty d\omega \left[a_\omega f_\omega + a_\omega^\dagger f_\omega^*\right] \tag{3.8}$$

Where the coefficients a_ω and $a_\omega^\dagger$ are interpreted as creation and annihilation operators on $\mathscr{I}^-$, as long as the modes are normalized with the KG inner product A.7, with $(f_\omega, f_{\omega'}) = \delta(\omega - \omega')$.

3.3.2 Cauchy Horizons

When talking about where particular modes of a field come from or where they are defined, it is often useful to talk about a *Cauchy surface.* A Cauchy surface is a surface which intersects every past and future causal curve. For time like curves it must only possible to intersect once. This is important for defining our fields as if we want to form a complete set of mode solutions, we need to be able to define their boundary conditions for all possible time-like and null paths. This condition is satisfied by a defining the boundary conditions on a Cauchy surface! As every time-like or null curve must intersect it.

3.3.3 Field Expansion In The Future

This understanding of Cauchy surfaces raises an issue for the defining of our scalar field φ in the far future. We are unable to simply use $\mathscr{I}^+$ to define the boundary conditions of our modes as we did with $\mathscr{I}^-$ as due to the presence of the black hole, $\mathscr{I}^+$ is not a Cauchy horizon as there are now time-like and null paths which enter the black hole and never end up on $\mathscr{I}^+$. This means in order to construct a proper basis we need to consider the Cauchy horizon $\mathscr{I}^+ \cap H^+$, where H^+ is the black hole horizon denoted with $r = 2M$ in 3.1.

It is then clear to us that it is the modes that end up on $\mathscr{I}^+$ are outgoing modes characterized by u and the modes that end up on H^+ are ingoing modes characterized by v. This means for the future expansion we need two sets of modes. Let us use p_ω to denote outgoing modes and q_ω for ingoing modes. With this we can expand our scalar field φ over this basis, once again using frequency ω as our integration variable:

$$\varphi = \int_0^\infty d\omega \left[b_\omega p_\omega + c_\omega q_\omega + b_\omega^\dagger p_\omega^* + c_\omega^\dagger q_\omega^*\right] \tag{3.9}$$

Since p_ω are the modes that propagate to $\mathscr{I}^+$, it is the expression of these modes in terms of the past modes on $\mathscr{I}^-$, f_ω and f_ω^*, that we are interested in. As these lead to the thermal emission spectrum. Since the p_ω modes are outgoing, they are characterized by having constant u. This means asymptotically on $\mathscr{I}^+$, these modes take the form:

$$p_\omega \propto \frac{1}{r\sqrt{\omega}} e^{-i\omega u} Y_{lm}$$

We will now go on to see how we can compare these modes to the modes of the past.

[6]Note that the fact that the star appears to come from a single point in the far past is an artifact of the conformal mapping.

3.3.4 Comparison Of u and v

The question we need to answer is how the vacuum in the far past, outside the collapsing body compares to the vacuum in the far future far from the collapsed black hole. We can recall that for the Unruh affect the way we quantify this is by calculating the expected particle number, which may not vanish for the annihilation operators of the far future. To check that this holds for the black hole case we need way of comparing the modes in the future with the modes in the past. To do this we follow what Hawking did as mentioned earlier. If we trace the path of a null geodesic, from $\mathscr{I}^+$ back to where it bounced of $r = 0$, and back out to $\mathscr{I}^-$. We can see that the geodesic started out as an incoming mode characterized by a certain value of v, but after bouncing of the center of $r = 0$ it becomes and outward traveling mode characterized by having a constant u. What we need then to know is how these values of v and u are related.

To do this the natural approach is to lean on our diagram in Figure 3.1. From this diagram it may appear that these distances are the same and that simply $u(v) - u(v_0) = v - v_0$ [7]. However, this will give us the wrong result as due to the different curvature of space-time in these regions we have to measure distances differently. To appropriately compare distance we need to measure the paths *affinely*. In principle we can parametrize a curve in space-time by any monotonically increasing function, however there is only a certain subset of these parameterizations that also preserve the geodesic equation that make these curves geodesics. The affine parametrization in the past far from the collapsing body may not be the same as that close to the black-hole's horizon.

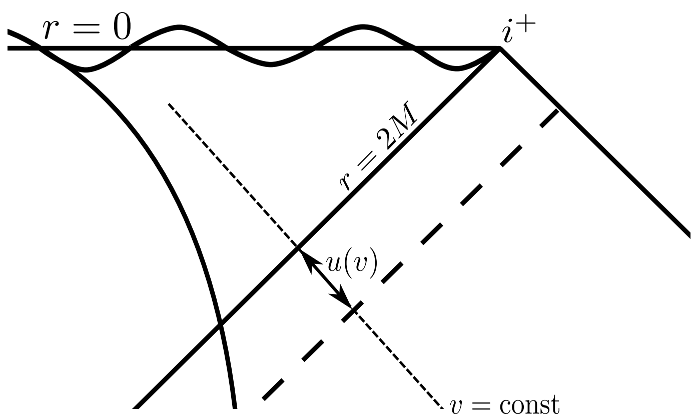


Figure 3.2: the

To find out what the two affine parameterizations are we need to solve the geodesic equation, in the two regions. The first region, in the future close to the black hole. For this calculation we can make use of the Eddington Finkelstein co-ordinates, as we are measuring the separation of $u(v)$ from the horizon, the line of measurement will be along an ingoing null geodesic with $v = \text{const}$. This is shown in Figure 3.2. This means in the co-ordinate system $x^\mu = (u, v, \theta, \phi)$ described in 1.2.7, we will have v, θ, ϕ constant and only u will vary. With this we can see that the tangent

[7] We can immediately see that this is wrong as although v_0 will be some finite value for the incoming geodesic, since $u(v_0)$ is on the horizon $r = 2M$, it cannot be finite, since $u(v_0)$ must blow up as $u(r = 2M) \to \infty$.

vector to our curve parametrized by the affine parameter λ is $V^\mu = \frac{dx^\mu}{d\lambda} = (\frac{du}{d\lambda}, 0, 0, 0)$ and satisfies:

$$\begin{aligned} \nabla_V V^\nu &= V^\mu \nabla_\mu V^\nu = 0 \\ \implies \frac{dx^\mu}{d\lambda}&\partial_\mu V^\nu + V^\nu \Gamma^\nu_{\mu\sigma} V^\sigma \\ =&\frac{d^2x^\nu}{d\lambda^2} + \frac{dx^\mu}{d\lambda}\Gamma^\nu_{\mu\sigma}\frac{dx^\sigma}{d\lambda} = 0 \end{aligned}$$

We can see that the since only $\frac{dx^0}{d\lambda} = \frac{du}{d\lambda}$ is non-vanishing, the only interesting one of these equations is for $\nu = 0$. This corresponds to:

$$\frac{d^2x^0}{d\lambda^2} + \frac{dx^\mu}{d\lambda}\Gamma^0_{\mu\sigma}\frac{dx^\sigma}{d\lambda} = \frac{d^2x^0}{d\lambda^2} + \frac{dx^0}{d\lambda}\Gamma^0_{00}\frac{dx^0}{d\lambda}$$

The Christoffel symbol $\Gamma^0_{00} = -\frac{1}{2}f'(r)$ is calculated in appendix B.2, where $f(r)$ is defined in 1.7. This means the geodesic equation results in the following differential equation for u, where we denote differentiation wrt λ with a dot: $\dot{u} = \frac{du}{d\lambda}$:

$$\ddot{u} - \frac{1}{2}f'(r)\dot{u}^2 = 0 \tag{3.10}$$

If we want to solve this ODE we need to be careful as r will depend on λ, so $f'(r)$ is not a constant along this geodesic. We can resolve this issue by finding some constants of the motion involving r. From our discussion in section B.4.3, we know that if we have a geodesic $x^\mu(\lambda)$ and the metric has a killing vector ξ, then the scalar quantity $\xi_\mu \frac{dx^\mu}{d\lambda}$, will be conserved. Since this is a scalar, this must hold for all co-ordinate systems.

We can for instance take the Schwarzschild metric 1.7, which clearly is independent of of the time co-ordinate t. This means as we discussed in section B.4.4, that the vector $\xi = \partial_t$ must be a killing vector of this metric. This means along our geodesic $x^\mu(\lambda)$, there is a constant:

$$\xi_\mu \frac{dx^\mu}{d\lambda} = \xi^\nu g_{\nu\mu}\frac{dx^\mu}{d\lambda} = -f(r)\frac{dt}{d\lambda} \equiv -E = \text{const}$$

Where we have used the fact that the components of ξ must be $\xi^\nu = \delta^\nu_t$. We also know that by definition this geodesic has $v = t + r^* = \text{const}$,which means we can write:

$$\begin{aligned} \frac{dv}{d\lambda} = \frac{dt}{d\lambda} + \frac{dr^*}{dr}\frac{dr}{d\lambda} &= \frac{dt}{d\lambda} + \frac{1}{f(r)}\frac{dr}{d\lambda} = 0 \\ \implies \frac{dr}{d\lambda} = -f(r)\frac{dt}{d\lambda} &= -E = \text{const} \end{aligned}$$

This is a useful result that will now let us solve this ODE 3.10. If we let $g = \frac{du}{d\lambda} = \dot{u}$ we can write 3.10 as:

$$\begin{aligned} &\ddot{u} - \frac{1}{2}f'(r)\dot{u}^2 = \frac{dg}{d\lambda} - \frac{1}{2}\frac{df}{dr}g^2 = 0 \\ \implies &\frac{dg}{d\lambda}\frac{1}{g^2} = \frac{1}{2}\frac{df}{d\lambda}\frac{d\lambda}{dr} \implies \int \frac{dg}{g^2} = -\frac{1}{2E}\int df \\ \implies &-\frac{1}{g} + c = -\frac{1}{2E}f(r) \implies \frac{du}{d\lambda} = \left(c - \frac{f(r)}{2E}\right)^{-1} \end{aligned} \tag{3.11}$$

We can fix this integration constant c by simply requiring that λ remain finite as we approach $r = 2M$, (clearly this must be the case if $dr/d\lambda = \text{const}$) this way since we know $u(r = 2M) = \infty$, then we must have that $\frac{du}{d\lambda}(r = 2M) = \infty$. This can only happen for $c = 0$ as then $f(r = 2M) = 0$ and the whole term goes to ∞ as needed. So we have $c = 0$.

We can use the fact that $dr/d\lambda = -E = \text{const}$ to write: $r - 2M = -E\lambda$, which fixes $\lambda(r = 2M) = 0$. This lets us write:

$$\frac{1}{f(r)} = \left(1 - \frac{2M}{r}\right)^{-1} = \left(1 - \frac{2M}{2M - E\lambda}\right)^{-1} = \frac{2M - e\lambda}{-E\lambda} = 1 - \frac{2M}{E\lambda}$$

With this we can see that the ODE 3.11 is reduced to:

$$\frac{du}{d\lambda} = -\frac{2E}{f(r)} = 2E - \frac{4M}{\lambda}$$
$$\implies u = 2E\lambda - 4M\ln\left(\frac{\lambda}{-C_1}\right)$$

Where we have absorbed the constant of integration into the logarithm. $C_1 > 0$, so that the argument of the logarithm is positive, as $\lambda < 0$ (this can be seen as we must have that $\lambda \to 0^-$ if $du/d\lambda$ is to go to $+\infty$ on $r = 2M$). In the region close to the black hole, $\lambda << 1 \implies u \approx -4M\ln\left(\frac{\lambda}{-C_1}\right)$. This means solving for the affine parameter, we have:

$$\lambda = -C_1 e^{-u/4M}$$

We can see that letting $C_1 = 1$ gives us the Kruskal co-ordinates we had in 1.2.8! Meaning these are the natural co-ordinates for describing geodesics near the event horizon.

In the second region in the past, far from the black hole, spacetime is approximately Minkowski. This means the metric 1.13 reduces to $ds^2 = -dudv + r^2 d\Omega^2$ so we can clearly see that in the co-ordinate system $x^\mu = (u, v, \theta, \phi)$ the Christoffel symbols with lower indices u or v will vanish as the derivatives of the metric vanish for any indice involving u or v. This means for the $\nu = 1 = v$ component, the geodesic equation, $\dot{x}^\mu \nabla_\mu \dot{x}^\nu = 0$, reads:

$$\dot{x}^\mu \nabla_\mu \dot{x}^1 = \dot{x}^\mu \partial_\mu \dot{x}^1 = \frac{dx^\mu}{d\lambda}\frac{\partial \dot{x}^1}{\partial x^\mu} = \frac{d^2 x^1}{d\lambda^2} = \frac{d^2 v}{d\lambda^2} = 0$$
$$\implies v = a\lambda + b$$

We can fix the constant b by requiring that the ingoing null geodesic v_0 which results in the outgoing geodesic that co-insides with the horizon $r = 2M$, and since in this region we fixed $\lambda = 0$ this means that $b = v_0$. With this we can finally we can equate the two affine parameters λ to get our relation between u and v:

$$\boxed{u(v) = -4M\ln\left(\frac{v_0 - v}{C_1 C_2}\right)} \tag{3.12}$$

Where we have relabeled the constant $a \to C_2 > 0$. We do not need to specify these constants as they do not impact the following calculation.

3.4 Comparison Of Modes

We now are in a position to compare our modes in the past, far from the black hole, with the modes in the future, close to the black hole, where outgoing particles appear to be created. Through 3.8 and 3.9 we have two sets of modes that we can expand our scalar field φ over. Then because these modes satisfy the same equation $\Box\varphi = 0$ and are orthonormal sets we can express the modes in terms of the other. This means the outgoing modes p_Ω in terms of the ingoing modes f_ω [8], via:

$$p_\Omega = \int_0^\infty d\omega' \left[\alpha_{\Omega\omega'} f_{\omega'} + \beta_{\Omega\omega'} f^*_{\omega'}\right]$$

Since these two modes have the same dependence on $1/r$ and Y_{lm}, as well as the same overall normalization factors, we can ignore these and just write down:

$$\frac{1}{\sqrt{\Omega}} e^{-i\Omega u} = \int_0^\infty \frac{d\omega'}{\sqrt{\omega'}} \left[\alpha_{\Omega\omega'} e^{-i\omega' v} + \beta_{\Omega\omega'} e^{i\omega' v}\right]$$

We can then multiply by $\frac{1}{2\pi} e^{\pm i\omega v}$ and integrate from $-\infty$ to ∞ we find in the plus case:

$$\frac{1}{2\pi}\int_{-\infty}^\infty dv \int_0^\infty \frac{d\omega'}{\sqrt{\omega'}} \alpha_{\Omega\omega'} e^{-iv(\omega\mp\omega')} = \int_0^\infty \frac{d\omega'}{\sqrt{\omega'}} \alpha_{\Omega\omega'} \delta(\omega \pm \omega') = \begin{cases} \frac{\alpha_{\Omega\omega}}{\sqrt{\omega}} \\ 0 \end{cases}$$
$$\frac{1}{2\pi}\int_{-\infty}^\infty dv \int_0^\infty \frac{d\omega'}{\sqrt{\omega'}} \beta_{\Omega\omega'} e^{iv(\omega\pm\omega')} = \int_0^\infty \frac{d\omega'}{\sqrt{\omega'}} \beta_{\Omega\omega'} \delta(-\omega \mp \omega') = \begin{cases} 0 \\ \frac{\beta_{\Omega\omega}}{\sqrt{\omega}} \end{cases}$$

Meaning we can concisely write:

$$\left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}} \int_{-\infty}^{v_0} dv e^{-i\Omega u} e^{\pm i\omega v}$$

Note that we have changed the upper bound of the integral from $\infty \to v_0$, as although the functions $\alpha_{\Omega\omega}, \beta\Omega\omega$ are in principle not functions of u and v, we know physically they will have a cut off relationship, where by they vanish for any geodesic with $v > v_0$, as these cannot escape the black hole. This change does not affect the above integrals as we only removed parts that were already 0. We can then use our relation between u and v from 3.12, to write:

$$\left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}} \int_{-\infty}^{v_0} dv e^{\pm i\omega v} e^{i4M\Omega \ln\left(\frac{v_0 - v}{C_1 C_2}\right)}$$

We can then make a change of variables to $s \equiv \mp(v - v_0)$:

$$\alpha_{\Omega\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{i\omega v_0} \int_0^\infty ds e^{-i\omega s} e^{i4M\Omega \ln\left(\frac{s}{C_1 C_2}\right)}$$
$$\beta_{\Omega\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{-i\omega v_0} \int_{-\infty}^0 ds e^{-i\omega s} e^{i4M\Omega \ln\left(\frac{-s}{C_1 C_2}\right)}$$

We can now analyze these integrals in the complex plane. For $\alpha_{\Omega\omega}$ we can see that the integrand has no poles in the lower complex plane, so we can take as a contour a quarter circle that goes from

[8] Note from here on, we use ω to denote the frequency of the incoming solutions and Ω for the frequency of the outgoing solutions.

0 to $+\infty$, then around in an arc to $-i\infty$ and back to 0. The integral around this closed contour must be 0 as there are no poles. Then seeing also that the integrand vanishes on the arc of the quarter circle as $e^{-i\omega s} = e^{-i\omega \mathrm{Re}(s)} e^{\omega \mathrm{Im}(s)}$, which vanishes as $\mathrm{Im}(s) \to -\infty$, the integral along the negative portion of the imaginary axis must be the same as (minus) the integral along the positive part of the real axis. This means we can makes the substitution $s' = is$ and get:

$$\alpha_{\Omega\omega} = -\frac{i}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{i\omega v_0} \int_{-\infty}^{0} ds' e^{\omega s'} e^{i4M\Omega \ln\left(\frac{is'}{C_1 C_2}\right)}$$

We can repeat the exact same procedure for $\beta_{\Omega\omega}$, this time however, the quarter circle is on the other side of the lower half complex plane. This is once again equal to the integral along the negative portion of the imaginary axis. resulting in:

$$\beta_{\Omega\omega} = \frac{i}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{-i\omega v_0} \int_{-\infty}^{0} ds' e^{\omega s'} e^{i4M\Omega \ln\left(\frac{-is'}{C_1 C_2}\right)}$$

We then have to deal with the complex valued logarithm. To do this we can pick the standard branch cut along the negative real axis, that is $\ln(z) = \ln|z| + i\theta$. This means:

$$\ln\left(\frac{is'}{C_1 C_2}\right) = \ln\left(\frac{|s'|}{C_1 C_2}\right) - \frac{i\pi}{2}$$
$$\ln\left(\frac{-is'}{C_1 C_2}\right) = \ln\left(\frac{|s'|}{C_1 C_2}\right) + \frac{i\pi}{2}$$

Meaning we can write:

$$\alpha_{\Omega\omega} = -\frac{i}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{i\omega v_0} e^{2M\Omega\pi} \int_{-\infty}^{0} ds' e^{\omega s'} e^{i4M\Omega \ln\left(\frac{|s'|}{C_1 C_2}\right)}$$
$$\beta_{\Omega\omega} = \frac{i}{2\pi}\sqrt{\frac{\omega}{\Omega}} e^{-i\omega v_0} e^{-2M\Omega\pi} \int_{-\infty}^{0} ds' e^{\omega s'} e^{i4M\Omega \ln\left(\frac{|s'|}{C_1 C_2}\right)}$$

From which we have the nice result that:

$$|\alpha_{\Omega\omega}|^2 = e^{8\pi M\Omega} |\beta_{\Omega\omega}|^2$$

This result can then be inserted into the frequency form of the relation B.18 (derived in appendix B.3.1), with $k = k'$ and thus $\Omega = \Omega'$:

$$\int_0^\infty d\omega \left[\alpha_{\Omega\omega}\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\beta^*_{\Omega'\omega}\right] = \delta(\Omega - \Omega')$$
$$\implies \int_0^\infty d\omega \left[|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2\right] = \int_0^\infty d\omega |\beta_{\Omega\omega}|^2 \left[e^{8\pi M\Omega} - 1\right] = \delta(0) \tag{3.13}$$

We show in appendix B.3.3 that the form of the expectation value of the number operator using Bogolubov transformations is just the integral of $|\beta_{\Omega\omega}|^2$ over ω. This means we can manipulate 3.13 to get:

$$\langle N_\Omega \rangle = \int_0^\infty d\omega |\beta_{\Omega\omega}|^2 = \frac{\delta(0)}{e^{8\pi M\Omega} - 1}$$

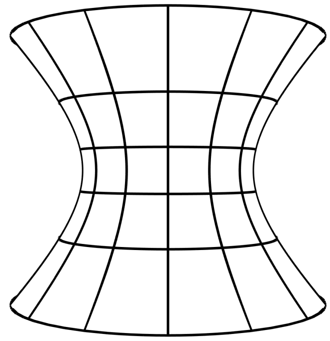
We can get rid of the $\delta(0)$ in the same manner that we did in section 2.6, by arguing that we have been considering all of space and this divergence is related to the volume of space considered. The physically meaningful quantity is the number density:

$$\boxed{\langle n_\Omega \rangle = \frac{\langle n_\Omega \rangle}{V} = \frac{1}{e^{8\pi M\Omega} - 1}} \tag{3.14}$$

This is once again the occupancy number for a Planck distribution with temperature:

$$\boxed{T = \frac{1}{8\pi M} = \frac{\kappa}{2\pi}}$$

Where $\kappa = 1/4M$ is the *surface gravity* of the Schwarzschild black hole.



Chapter 4

Anti-Evaporation

4.1 Introduction

We will now turn our attention to Raphael Bousso and Stephen W. Hawking's 1997 paper titled "(Anti-)Evaporation of Schwarzschild-de Sitter Black Holes" [3]. As the name suggests this paper builds on Hawking's previous work on the evaporation of black holes by analyzing the stability of a specific type of black hole, namely a Schwarzschild de Sitter black hole. This is just a regular Schwarzschild black hole immersed in a de Sitter background. In this space time it can be shown that the metric gives rise to two horizons, one is the original black hole horizon and the other is a cosmological horizon that arises due to the expansion of the universe.

Bousso and Hawking study in particular, the scenario where the black hole horizon is near its maximal mass meaning the black hole horizon is close to the same size as the cosmological horizon. The degenerate solution where the two horizon are the same is known as the Nariai solution. In the limit where the black hole is slightly smaller then the cosmological horizon Bousso and Hawking studied the evolution of the black hole horizon due to quantum effects. Their model includes the one-loop effective action in the s-wave and large N approximation. Meaning they add effective action terms to incorporate the quantum effects due to the presence of a large number of fields. In analyzing the stability of the black hole, they find that when perturbed from the maximal size there is in fact a stable mode that returns the black hole to the Nariai point, hence the title anti-evaporation. However, the presence of other modes that are not stable leads them to conclude that this solution is overall unstable to evaporation.

4.2 Schwarzschild de Sitter Metric

We now look at the aforementioned Schwarzschild de Sitter metric, which is is a combination of the two metrics, 1.7 and 4.1. This is the most general spherically symmetric metric which satisfies the Einstein Hilbert action with a cosmological constant 1.18. The metric takes the form:

$$ds^2 = -\left(1 - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2}dr^2 + r^2 d\Omega_2 \tag{4.1}$$

Here μ is a mass parameter [1] that controls the size of the black hole. We can see that μ can range from $0 < \mu < \frac{1}{3\sqrt{\Lambda}}$, with this upper bound corresponding to the degenerate Nariai limit.

To parametrize the behavior of the solution close to the Nariai limit, Bousso and Hawking followed Ginsparg and Perry [11] in writing the following co-ordinates. First they parametrized μ by $9\mu^2\Lambda = 1 - 3\epsilon^2$, then defined the co-ordinates:

$$t = \frac{1}{\epsilon\sqrt{\Lambda}}\psi, \quad r = \frac{1}{\sqrt{\Lambda}}\left[1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right] \tag{4.2}$$

In these co-ordinates the metric can be shown (see appendix C.1) to take the following form for first order in ϵ:

$$-\frac{1}{\Lambda}\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi d\psi^2 + \frac{1}{\Lambda}\left(1 - \frac{2}{3}\epsilon\cos\chi\right)d\chi^2 + \frac{1}{\Lambda}\left(1 - 2\epsilon\cos\chi\right)d\Omega_2 \tag{4.3}$$

This metric 4.3 is obtained by expanding up to cubic order in epsilon. We show in appendix C.1 that at this order $f(r) = \sin^2\psi\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\epsilon^2$, so we can see there are two horizons corresponding to $\chi = 0$, which is the black hole horizon (as then $r < \frac{1}{\sqrt{\Lambda}}$) and $\chi = \pi$ which is the cosmological horizon (as then $r > \frac{1}{\sqrt{\Lambda}}$).

4.2.1 Two Dimensional Model

In their paper Bousso and Hawking reduce a four dimensional model to a two dimensional model. This is based on previous papers in which simple 1 + 1-d black hole models had been studied with the introduction of evaporation [5] [17] [17]. Bousso and Hawking obtain their two dimensional model by starting with the regular 4-dimensional Einstein-Hilbert action 1.18 (with the addition of scalar fields), before integrating out the angular variables to obtain a 2-d action. In using this action Bousso and Hawking make a spherically symmetric ansatz for the metric that takes the following form:

$$ds^2 = e^{2\rho}\left[-dt^2 + dx^2\right] + e^{-2\phi}d\Omega_2 \tag{4.4}$$

Our goal in this section will be to motivate the choice of this metric based on the expansion near the maximal mass given by the metric 4.3.

Embedding Co-ordinates

In the maximal black hole case $\epsilon = 0$ [2], here the metric 4.3 reduces to[3]:

$$ds^2 = \frac{1}{\Lambda}\left[-\sin^2\chi d\psi^2 + d\chi^2 + d\Omega_2\right] \tag{4.5}$$

We can then proceed to manipulate this metric further by defining $z = \cos\chi \implies \chi = \arccos(z)$ then we have that:

$$d\chi = \frac{dz}{\sqrt{1 - z^2}} \implies d\chi^2 = \frac{dz^2}{1 - z^2}$$

[1] All though it is not straighforward to define mass in asymtopically de Sitter space times, it can be shown that this parameter μ is exactly mass [8].

[2] We should note that we are not fully setting $\epsilon \to 0$ here as this would make the new time co-ordinate $\psi \to \infty$. This is better thought of as taking the metric to the next leading order, i.e. ignoring any contributions of $\mathcal{O}(\epsilon)$.

[3] If we were to make the time co-ordinate euclidean, i.e. $\psi_E = i\psi$, then this metric becomes exactly $S^1 \times S^2$.

And $\sin\chi = \sqrt{1-z^2}$, so we can write 4.5 as:

$$ds^2 = -\frac{1}{\Lambda}(1-z^2)d\psi^2 + \frac{1}{\Lambda}\frac{dz^2}{1-z^2} + \frac{1}{\Lambda}d\Omega_2 \tag{4.6}$$

This looks *very* like the de Sitter space metric 4.1, except for the lack of a z^2 term in front of $d\Omega_2$. We can still use the fact that this is similar to the de Sitter metric to write the first two co-ordinates as an embedding in a 3-d space, as we discussed in 1.3.2. If we define:

$$X_0 = \sqrt{\frac{1-z^2}{\Lambda}}\sinh\psi, \quad X_1 = \sqrt{\frac{1-z^2}{\Lambda}}\cosh\psi, \quad X_2 = \frac{z}{\Lambda} \tag{4.7}$$

With these $X_\mu X^\mu = \frac{1}{\Lambda}$ and it can be shown (see Appendix C.1.1) that $dX_\mu dX^\mu$ matches the first two terms of 4.6.

Global Co-ordinates

Since this places our first two co-ordinates into an embedding like de Sitter spacetime, we can then proceed to use the transformations that apply to de Sitter space time. Following Hartman in [12] we can define what are known as *global co-ordinates*, which are called global as they cover the entire hyperbola. These are defined as:

$$X_0 = \frac{1}{\sqrt{\Lambda}}\sinh(\sqrt{\Lambda}T), \quad X_1 = \frac{1}{\sqrt{\Lambda}}\cosh(\sqrt{\Lambda}T)\tilde{y}_1, \quad X_2 = \frac{1}{\sqrt{\Lambda}}\cosh(\sqrt{\Lambda}T)\tilde{y}_2$$

Where here $\tilde{y}_i, \quad i=1,2$ are co-ordinates on the one sphere S^1, i.e. $\tilde{y}_1 = \sin x, \quad \tilde{y}_2 = \cos x$ so that $\sum_i \tilde{y}_i^2 = 1$ which implies $\sum_i \tilde{y}_i d\tilde{y}_i = 0$. These co-ordinates clearly satisfy the necessary condition $X_\mu X^\mu = \frac{1}{\Lambda}$ and their differentials are given by:

$$dX_0 = \cosh\left(\sqrt{\Lambda}T\right)dT, \quad dX_i = \sinh(\sqrt{\Lambda}T)\tilde{y}_i dt + \frac{1}{\sqrt{\Lambda}}\cosh(\sqrt{\Lambda}T)d\tilde{y}_i$$

Which means this part of the metric is:

$$\begin{aligned}-dX_0^2 + dX_1^2 + dX_2^2 &= -\cosh^2(\sqrt{\Lambda}T)dT^2 + \sinh^2(\sqrt{\Lambda}T)\underbrace{(\tilde{y}_1^2+\tilde{y}_2^2)}_{1}dT^2 \\ &\quad + \frac{2}{\sqrt{\Lambda}}\sinh(\sqrt{\Lambda}T)\cosh(\sqrt{\Lambda}T)\cancelto{0}{(\tilde{y}_1 d\tilde{y}_1 + \tilde{y}_2 d\tilde{y}_2)} + \frac{1}{\Lambda}\cosh^2(\sqrt{\Lambda}T)\underbrace{(d\tilde{y}_1^2 + d\tilde{y}_2^2)}_{dx^2} \\ &= -dT^2 + \frac{1}{\Lambda}\cosh^2(\sqrt{\Lambda}T)dx^2\end{aligned} \tag{4.8}$$

Conformal Co-ordinates

With this we are almost at the form of 4.4, what is left is to make these two co-ordinates conformally flat. We can achieve this from the above global co-ordinates in the following way. Again following Hartman we can define the following co-ordinates:

$$\cosh(\sqrt{\Lambda}T) = \frac{1}{\cos t} \implies \sqrt{\frac{1}{\cos^2 t} - 1} = \sinh(\sqrt{\Lambda}T)$$

Which means:

$$\sqrt{\Lambda}\frac{dT}{dt}\sinh(\sqrt{\Lambda}T) = \frac{\sin t dt}{\cos^2 t} \implies dT = \frac{\sin t}{\sqrt{\Lambda}\cos^2 t}\frac{dt}{\sqrt{\frac{1}{\cos^2 t} - 1}} = \frac{dt}{\sqrt{\Lambda}\cos t}$$

Plugging this in to the global co-ordinate metric 4.8 we get:

$$-dT^2 + \frac{1}{\Lambda}\cosh^2(\sqrt{\Lambda}T)dx^2 = \frac{1}{\Lambda\cos^2 t}\left[-dt^2 + dx^2\right]$$

Since we have shown this 2d metric is the same as the first two terms in 4.6 we can finally write 4.6 fully as:

$$ds^2 = \frac{1}{\Lambda\cos^2 t}\left[-dt^2 + dx^2\right] + \frac{1}{\Lambda}d\Omega_2 \tag{4.9}$$

This is the exact form of 4.4, and we can see that near the maximal point we have that:

$$e^{2\rho} = \frac{1}{\Lambda\cos^2 t}, \quad e^{-2\phi} = \frac{1}{\Lambda} \tag{4.10}$$

This will be useful as a starting point for introducing perturbations later. But for now we will keep the exponentials.

Relation Between χ and x

We should also notice that in the above co-ordinate system x is the co-ordinate on the one sphere. This is because x is the angle that we wrote our co-ordinates $\tilde{y}_i$ on the one sphere in terms of. In fact if we write down both the co-ordinate transformations involving the embedding co-ordinate X_1, we see that:

$$X_1 = \sqrt{\frac{1-z^2}{\Lambda}}\cosh\psi, \quad X_1 = \frac{1}{\sqrt{\Lambda}}\cosh(\sqrt{\Lambda}T)\tilde{y}_1$$

$$\implies \sin\chi\cosh\psi = \cosh\sqrt{\Lambda}T\sin x$$

$$\implies \sin\chi = a\sin x, \quad a > 0$$

Where we have remembered that $z = \cos z \implies \sqrt{1-z^2} = \sin\chi$ and that $\tilde{y} = \sin x$. Since $\cosh\psi, \cosh(\sqrt{\Lambda}T) > 1$ this means χ and x are closely related and more importantly have the same value at the horizons at $t = 0$ as then $a = 1$. The black hole horizon located at $\chi = 0$ has $\sin(\chi = 0) = 0 \implies \sin x = 0$ must also corresponds to $x = 0$ and the cosmological horizon at $\chi = \pi \implies \sin(\chi = \pi) = 0$ must also correspond to $x = \pi$ as x must have increased and π is the next possible root of the sin function.

We are now ready to use this metric to write down the action.

4.3 Classical Action

4.3.1 Dimension Reduction

As stated earlier Bousso and Hawking start with the 4-dimensional Einstein Hilbert action with a cosmological constant 1.18 with the addition of N different free scalar fields denoted f_i. Being free fields they only have a kinetic term in the Lagrangian so the action takes the following form:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right] \tag{4.11}$$

Here $(\nabla f_i)^2 = g^{\mu\nu}(\nabla_\mu f_i)(\nabla_\nu f_i)$. Many different scalar fields are introduced as later it is useful to take some limits involving large N. As discussed in section 4.2.1 an appropriate ansatz for the 4d metric is 4.4. The next step is to integrate out the spherical co-ordinates on the 2-sphere, to do this we first need to notice that the only part of the above action 4.11 that depends on the angular co-ordinates θ and ϕ is the determinant of the metric g. From 4.4 we can read off that the determinant of g is:

$$g \equiv \det(g_{\mu\nu}) = -e^{4\rho}e^{-4\phi}\sin^2\theta$$

This means that the angular part of the above action integral is:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = 4\pi$$

It appears however that in order to make their model more akin to the 2d models Bousso and Hawking rescale the gravitational constant to account for this 4π so that the action still has an overall factor of $1/16\pi$. This is what the factor would be if a $2d$ model was considered from the start. To make it clear when we are talking about the 2-d metric and the 4-d metric we will place a $\sim$ on any 2-d quantities.Once the angular co-ordinates have been integrated out we are left with only the first two components of the metric 4.4:

$$d\tilde{s}^2 = e^{2\rho}\left[-dt^2 + dx^2\right] \tag{4.12}$$

This metric has an $2d$ determinant: $\tilde{g} = -e^{4\rho} \implies \sqrt{-g} = \sqrt{-\tilde{g}}\sin\theta e^{-2\phi}$. Also in reducing the action form 4-d to 2t-d the value of the Ricci scalar R changes to $\tilde{R}$. We show in appendix via Mathematica that for the two metrics 4.4 and 4.12 these Ricci scalars (R and $\tilde{R}$ respectively) are:

$$\begin{aligned} R &= 2e^{2\phi} + 6e^{-2\rho}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] + 4e^{-2\rho}\left[\partial_x^2\phi - \partial_t^2\phi\right] + 2e^{-2\rho}\left[\partial_t^2\rho - \partial_x^2\rho\right] \\ \tilde{R} &= 2e^{-2\rho}\left[\partial_t^2\rho - \partial_x^2\rho\right] \\ \implies R &= \tilde{R} + 2e^{2\phi} - 6(\tilde{\nabla}\phi)^2 - 4e^{-2\rho}\left[\partial_t^2\phi - \partial_x^2\phi\right] \end{aligned} \tag{4.13}$$

Where we have used the fact that since in the metric 4.12: $e^{2\rho} = g_{tt} = g_{xx} \implies g^{tt} = g^{xx} = e^{-2\rho}$ (as the metric is diagonal) which implies that $(\tilde{\nabla}\phi)^2 = g^{\mu\nu}(\partial_\mu\phi)(\partial_\mu\phi) = e^{-2\rho}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right]$.

With these considerations the action 4.11 can be written as follows after integrating out the angular co-ordinates:

$$S = \frac{1}{16\pi}\int d^2x\sqrt{\tilde{g}}e^{-2\phi}\left[\tilde{R} + 2e^{2\phi} - 6(\tilde{\nabla}\phi)^2 - 4e^{-2\rho}\left[\partial_t^2\phi - \partial_x^2\phi\right] - 2\Lambda - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2\right] \tag{4.14}$$

Where we can notice that we don't have any change to the scalar fields as we are not interested in any angular effects of these fields so we can assume they are all in the first spherical harmonic (the only spherically symmetric one) such that the derivative wrt to θ and ϕ vanish. This is waht is meant by the s wave approximation, as for low energy scattering the terms in the s orbital are the only relevant terms.

We can deal with the middle term in this action in the following way. Writing down only this term and remembering that $\sqrt{\tilde{g}} = e^{2\rho}$ we have:

$$I \equiv \frac{1}{16\pi}\int d^2x\sqrt{\tilde{g}}e^{-2\phi}\left[-4e^{-2\rho}\left[\partial_t^2\phi - \partial_x^2\phi\right]\right] = -\frac{1}{4\pi}\int d^2xe^{-\phi}\left[\partial_t^2\phi - \partial_x^2\phi\right]$$

We can then integrate each of these terms by parts picking up a minus sign:

$$\begin{aligned} I &= \frac{1}{4\pi}\int d^2x\left[(\partial_t\phi)\partial_t(e^{-2\phi}) - (\partial_x\phi)\partial_x(e^{-2\phi})\right] \\ &= -\frac{2}{4\pi}\int d^2x\left[\partial_t^2\phi - \partial_x^2\phi\right]e^{-2\phi} = -\frac{8}{16\pi}\int d^2xe^{2\rho}e^{-2\rho}\left[\partial_t^2\phi - \partial_x^2\phi\right]e^{-2\phi} \\ &= \frac{1}{16\pi}\int d^2x\sqrt{\tilde{g}}e^{-2\phi}\left[8(\tilde{\nabla}\phi)^2\right] \end{aligned}$$

Where we assume that the fields vanish at infinity.

Plugging this into the action 4.14 we have the 2-dimensional action that Bousso and Hawking use:

$$S = \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}e^{-2\phi}\left[\tilde{R} + 2e^{2\phi} + 2(\tilde{\nabla}\phi)^2 - 2\Lambda - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2\right] \tag{4.15}$$

We can now notice that the the parameter ϕ which controls the radius of the two sphere (as $r = e^{-\phi}$) now has the form of the kinetic part of a scalar field. This scalar field is known as a dilaton field and appears in many instances where certain co-ordinates are integrated out, as is done in many compactification schemes in string theory.

4.4 Effective Action

In this section we discuss the motivation for Bousso and Hawking's addition of an effective action term to the action 4.15. This addition is an alternate way of adding quantum effects such as evaporation to our model. We saw in Chapter 3 that a semi-classical treatment of a scalar field in the presence of a black hole gives rise to radiation, but the derivation we used made the approximation that the black hole mass stayed constant through out the evaporation. This model had no way of accounting for back reaction [4] and in turn was only able to posit that the mass of the black hole decreased because energy was leaving the system. The approach we outline below introduces quantum effects through the energy momentum tensor, which ensures that evaporation is associated with a flow of energy, allowing for a description involving a back reaction.

4.4.1 Conformal anomaly

If we want to introduce quantum effects to our action in then instead of using the classical $T_{\mu\nu}$, we should instead deal with its expectation value $\langle T_{\mu\nu}\rangle$. When we quantize fields, like the ones already

[4] Note by back reaction we mean the process by which evaporation actually changes the background geometry of the spacetime, by say changing the mass of the black hole.

introduced in our action, we formally find that the value of $\langle T_{\mu\nu}\rangle$ is divergent. Hence there is a need for some sort of re-normalization scheme to get rid of these infinities while obtaining finite and physically meaningful results. There are several possible re-normalization schemes, we will see that in an earlier paper [2] Bousso and Hawking use a method known as zeta function method. It turns out that when we re-normalize the expectation value of the stress energy tensor in this way, the process introduces a non vanishing trace to the stress energy tensor; $g^{\mu\nu}T_{\mu\nu} \neq 0$. Since this was a quantity that vanished classically (as is showed in Appendix C.2.1), we call this non-vanishing term a *trace anomaly*. Since the trace of $T_{\mu\nu}$ was expected to vanish due to conformal symmetry, we call this particular anomaly a conformal anomaly.

4.4.2 Path Integral

In order to to properly include the description of evaporation involving back reaction in our model a re-normalized stress energy tensor must be included. In [7] it was shown that the amount of radiation at infinity is proportional to the trace anomaly. Hence this is something we need to calculate. If we want to calculate the expectation value of the stress energy tensor as part of this radiation we cannot simply use the definition C.3 with the classical action. To have a quantum expectation value we need to consider all possible field configurations weighted by some factor that determines how likely those configurations are. This is given through the path integral formulation [5]. In this formalism the integral is over all possible configurations of the fields, the bare integral times the weight factor can be used as a generating functional. Denoted Z; it is defined for a field φ as follows:

$$Z = \int \mathcal{D}[\varphi] e^{iS[\varphi]} \tag{4.16}$$

Where S is the classical action. The path integral 4.16 leads to a natural interpretation of the classical limit of quantum mechanics, which we outline in Appendix C.2.2.

4.4.3 Effective Action

We can then consider defining a quantity Γ such that $e^{i\Gamma} = Z$. Taking the functional derivative of Z with respect to $g^{\mu\nu}$ then results in the following:

$$\begin{aligned}\frac{\delta Z}{\delta g^{\mu\nu}} &= \int \mathcal{D}[\varphi] i \frac{\delta S}{\delta g^{\mu\nu}} e^{iS[\varphi]} \\ &= \frac{\delta}{\delta g^{\mu\nu}} \left(e^{i\Gamma}\right) = i \frac{\delta \Gamma}{\delta g^{\mu\nu}} e^{i\Gamma}\end{aligned}$$

Re-arranging this we have:

$$\begin{aligned}\frac{\delta \Gamma}{\delta g^{\mu\nu}} &= \frac{\int \mathcal{D}[\varphi] \frac{\delta S}{\delta g^{\mu\nu}} e^{iS[\varphi]}}{\int \mathcal{D}[\varphi] e^{iS[\varphi]}} \\ \implies -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}} &= \frac{\int \mathcal{D}[\varphi] - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} e^{iS[\varphi]}}{\int \mathcal{D}[\varphi] e^{iS[\varphi]}} = \frac{\int \mathcal{D}[\varphi] T_{\mu\nu} e^{iS[\varphi]}}{\int \mathcal{D}[\varphi] e^{iS[\varphi]}}\end{aligned} \tag{4.17}$$

[5] For a formal introduction to path integral formulation of quantum field theory see chapter 9 of Peskin & Schroeder

This can then be recognized as a good definition of $\langle T_{\mu\nu}\rangle$, since it takes into account the value of $T_{\mu\nu}$ for each field configuration, normalized by the total amount of these configurations. Furthermore, looking at the LHS of 4.17, we can see that this takes the exact same form as the classical definition of the stress energy tensor C.3. Except now the classical action S is replaced by this new quantity Γ. In this regard Γ is called the *effective action* as it can be used to calculate the quantum expectation value of $\langle T_{\mu\nu}\rangle$. With this, the procedure for including quantum effects in the action becomes clear, we simply add to our classical action this effective action, then the evaporation and back reaction effects will be automatically included in the model. There is one caveat to this, when this particular effective action in SI units is proportional to $\hbar$, which is negligible in the classical limit $\hbar \to 0$. This makes sense as the effects of evaporation are very small. One way to make the effects of evaporation larger is to have N scalar fields, then N effective action terms will not vanish if N is large enough to keep $N\hbar$ fixed.

4.4.4 Calculation of Effective Action

From the action 4.15 we can see that after the dimension reduction to a 2-d system the action for each of the N scalar fields f takes the form:

$$S_f = \frac{1}{32\pi}\int d^2x\sqrt{-\tilde{g}}e^{-2\phi}(\tilde{\nabla}f)^2 \tag{4.18}$$

Since the scalar fields were introduced in the 4-d action 4.11 they noticeably picked up a coupling to the dilaton field ϕ, through $e^{-2\phi}$. This is something that does not happen in models that start out as 2-d and introduce scalar fields from there. This action S_f can be written in a more appealing way. Note that $(\tilde{\nabla}f)^2 = \tilde{\nabla}_\mu\left(f\tilde{\nabla}^\mu f\right) - f\tilde{\nabla}^2 f$, hence the action becomes:

$$\begin{aligned} S_f &= \frac{1}{32\pi}\int d^2x\sqrt{-\tilde{g}}e^{-2\phi}\left[\tilde{\nabla}_\mu\left(f\tilde{\nabla}^\mu f\right) - f\tilde{\nabla}^2 f\right] \\ &= \frac{1}{32\pi}\int d^2x\sqrt{-\tilde{g}}\left[\tilde{\nabla}_\mu\left(e^{-2\phi}f\tilde{\nabla}^\mu f\right) - \left(\tilde{\nabla}_\mu e^{-2\phi}\right)\left(f\tilde{\nabla}^\mu f\right) - e^{-2\phi}f\tilde{\nabla}^2 f\right] \end{aligned}$$

This first term is then just a total derivative, so we are left with:

$$S_f = \int d^2x\sqrt{-g}f\hat{D}f$$

Where:

$$\hat{D} = \frac{1}{32\pi}e^{-2\phi}\left[2\left(\tilde{\nabla}_\mu\phi\right)\tilde{\nabla}^\mu - \tilde{\nabla}^2\right]$$

This form of the action is useful as it is possible to carry out integrals of the form $\int\mathcal{D}[f]e^{-S_f}$, when the action is quadratic in the fields f. To carry out the integral the operator $\hat{D}$ is diagonalised and after a change of variables what is left over is just a product of Gaussian integrals over the eigenvalues of the operator $\hat{D}$. This evaluates to a product of the eigenvalues which is nothing more then the determinant of the operator $\hat{D}$. For more details on this calculation see pg 189 of [15]. This all means the effective action is related to the operator $\hat{D}$ via:

$$e^{i\Gamma} = \int\mathcal{D}[f]e^{-\int d^2x\sqrt{-g}f\hat{D}f} \sim (\det\hat{D})^{\#} = e^{\#\ln\det\hat{D}} = e^{\#\mathrm{tr}[\ln\hat{D}]}$$

Where in the last line we have used the identity $\ln\det(\hat{D}) = \mathrm{tr}[\ln\hat{D}]$. This identifies $\Gamma \sim \mathrm{tr}[\ln\hat{D}]$, which in principle allows the calculation of Γ though this is not straightforward. Since computing this result exactly is difficult, it is common to expand $\mathrm{tr}[\ln\hat{D}]$ in a Taylor series and analyse the quantum corrections term by term, just like the Feynman diagram approach to QFT. This is why the action Bousso and Hawking use is called the one-loop effective action as they expand to the one loop level.

Bousso and Hawking compute the one loop effective action for dilaton coupled scalars in [2]. They do this by noticing that the trace anomaly they are trying to add to their model must only consist of co-variant terms with two metric derivatives. There are only three quantities for the action they are considering that satisfy this, meaning:

$$T = q_1\tilde{R} + q_2(\tilde{\nabla}\phi)^2 + q_3\Box\phi$$

Where the q_i are undetermined coefficients. In [2] Bousso and Hawking consider several scenarios in order to calculate these coefficients, where they find $q_1 = \frac{1}{24\pi}$, $q_2 = -\frac{1}{4\pi}$ and $q_3 = -\frac{1}{12\pi}$. With this all is left to do is find the effective action that gives rise to this trace anomaly (i.e. through 4.17). It can be shown that this is the following effective action:

$$W^* = -\frac{1}{48\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R}\frac{1}{\Box}\tilde{R} - 6(\tilde{\nabla}\phi)^2\frac{1}{\Box}\tilde{R} - 2\phi\tilde{R}\right]$$

Bousso and Hawking then add this N copies this effective action to the classical action, one for each of the scalar fields f considered. The middle term containing $6(\tilde{\nabla}\phi)^2\frac{1}{\Box}\tilde{R}$ is not carried through the calculation by Bousso and Hawking as it does not contribute to the final equation 4.37. [6]

$$\begin{aligned} S_{\mathrm{tot}} &= \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[e^{-2\phi}\left(\tilde{R} + 2e^{2\phi} + 2(\tilde{\nabla}\phi)^2 - 2\Lambda - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2\right) - \frac{N}{3}\left(\frac{1}{2}\tilde{R}\frac{1}{\Box}\tilde{R} - 2\phi\tilde{R}\right)\right] \\ &= \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\left(e^{-2\phi} - \frac{N}{6}\left(\frac{1}{\Box}\tilde{R}\right) + \frac{2N}{3}\phi\right)\tilde{R} + 2 + 2e^{-2\phi}(\tilde{\nabla}\phi)^2 - 2\Lambda e^{-2\phi} - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2\right] \end{aligned}$$

Defining $\kappa = \frac{2N}{3}$ and $w = 2$, integrating out the f fields:

$$S_{\mathrm{tot}} = \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\left(e^{-2\phi} - \frac{\kappa}{4}\left(\frac{1}{\Box}\tilde{R}\right) + \frac{\kappa}{2}w\phi\right)\tilde{R} + 2 + 2e^{-2\phi}(\tilde{\nabla}\phi)^2 - 2\Lambda e^{-2\phi}\right] \quad (4.19)$$

4.4.5 Rendering the Action Local

The action 4.19 with the addition of the effective action terms, contains the greens function of the $\Box$ operator, which is not a local function as it contains in its Taylor expansion infinitely many derivatives of the spatial co-ordinates [7]. Containing infinitely many derivatives makes this function non-local as it needs infinitely many initial conditions, which amounts to needing non local information [8]. In order to render this action local Bousso and Hawking followed Hayward in

[6] This is because it enters the equation of motion as a $\partial^2\phi$ term and as can be seen from equation C.7, this term is always of order $\mathcal{O}(\epsilon^2)$.

[7] Recall that the greens function of $\Box$ is $\frac{1}{\Box} \sim \frac{1}{|x-x'|}\delta(t - t' - \frac{1}{c}|x - x'|)$

[8] Think of a lattice where the nth point away from the current point adds an initial condition for the nth derivative, in this way infinity many derivatives requires you to be non-locally far away.

[14] in introducing a new scalar field Z that mimics these non local terms. It does this by having its equation of motion be such that when they are subbed back into the Lagrangian, result exactly in these non-local terms.

In the above action 4.19 the only non-local term is $\left(\frac{1}{\Box}\tilde{R}\right)$. To render this local consider the following addition to the action 4.19:

$$\begin{aligned} S_Z &= \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{\kappa}{2}Z\tilde{R} - \frac{\kappa}{4}(\tilde{\nabla}Z)^2\right] = \frac{1}{16\pi}\int d^2x\left[e^{2\rho}\frac{\kappa}{2}Z\tilde{R} - \frac{\kappa}{4}\left[(\partial_x Z)^2 - (\partial_t Z)^2\right]\right] \\ &= \frac{1}{16\pi}\int d^2x\left[e^{2\rho}\frac{\kappa}{2}Z\tilde{R} + \frac{\kappa}{4}\left[\partial_x^2 Z - \partial_t^2 Z\right]Z\right] = \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{\kappa}{2}Z\tilde{R} + \frac{\kappa}{4}\left(\Box Z\right)Z\right] \end{aligned} \tag{4.20}$$

Where we have integrated by parts the second term and used the fact that $e^{-2\rho}\left[\partial_x^2 Z - \partial_t^2 Z\right] = \Box Z$. In equation 4.25 we show that varying this action wrt Z results in the equation of motion $\Box Z = 2e^{-2\rho}\left[\partial_x^2\rho - \partial_t^2\rho\right] = -\tilde{R}$. This means we can use the greens function of $\Box$ to solve for Z. By the definition of the integral operator $\frac{1}{\Box}$:

$$Z = -\frac{1}{\Box}\tilde{R}$$

Using this we can write our S_z action 4.20 as:

$$\begin{aligned} S_Z &= \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{\kappa}{2}Z\tilde{R} - \frac{\kappa}{4}\tilde{R}Z\right] = \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{\kappa}{4}Z\tilde{R}\right] \\ &= \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[-\frac{\kappa}{4}\frac{1}{\Box}\tilde{R}\right] \end{aligned}$$

This is exactly the non local term that appears in the action 4.19, so we can safely replace it with our ansatz for the Z action, the first term in 4.20. With this the total action is now:

$$S_{\text{tot}} = \frac{1}{16\pi}\int d^2x\sqrt{-\tilde{g}}\left[\left(e^{-2\phi} - \frac{\kappa}{2}\left(Z + w\phi\right)\right)\tilde{R} - \frac{\kappa}{4}(\tilde{\nabla}Z)^2 + 2 + 2e^{-2\phi}(\tilde{\nabla}\phi)^2 - 2\Lambda e^{-2\phi}\right] \tag{4.21}$$

4.5 Equations of Motion

Now that we have obtained a non-local action we can begin to vary the fields and find the equations of motion. For this we can use the curved spacetime Lagrange density equations we found in B.3. Since all the fields we are dealing with scalar fields we can just replace the co-variant derivative with partial derivatives in this equation. [9] This means for some scalar field φ our EoM will take the following form:

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0 \tag{4.22}$$

Some Notation

We would quickly like to define some notation that will allow us to write cleaner expressions in the following equations of motion. We follow Hawking and Bousso in denoting derivatives wrt t with

[9] This is because by definition of the co-variant derivative, for a scalar function f: $\nabla_\mu f = \partial_\mu f$.

a dot (i.e. $\partial_t \phi = \dot{\phi}$) and derivatives wrt x with a prime (i.e. $\partial_x \phi = \phi'$). Furthermore we define the following shorthand:

$$\partial f \partial g = -\dot{f}\dot{g} + \dot{f}\dot{g}, \quad \partial^2 f = -\dot{f}^2 + f''$$

These are essentially just contractions of the partial derivatives with the 2-d Minkowski metric, so we will often denote intermediate steps of the calculations involving these terms with the Minkowski metric $\eta_{\mu\nu}$.

It will also be useful to define:

$$\delta f \delta g = \dot{f}\dot{g} + f'g', \quad \delta^2 f = \ddot{f} + f''$$

4.5.1 Lagrangian Density

Before we can find the EoM, we need to properly write down our Lagrangian. If we look at the action 4.21 and remember that $\sqrt{-\tilde{g}} = e^{2\rho}$ and that per 4.13 $\tilde{R} = -2e^{-2\rho}\partial^2\rho$, then the Lagrange density is:

$$\mathcal{L} = e^{2\rho}\left[\left(e^{-2\phi} - \frac{\kappa}{2}(Z + w\phi)\right)(-2e^{-2\rho}\partial^2\rho) - \frac{\kappa}{4}(\tilde{\nabla}Z)^2 + 2 + 2e^{-2\phi}(\tilde{\nabla}\phi)^2 - 2\Lambda e^{-2\phi}\right]$$

If we then remember that for any field φ: $(\tilde{\nabla}\varphi)^2 = g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) = e^{-2\rho}\left[(\partial_x\varphi)^2 - (\partial_t\varphi)^2\right] = e^{-2\rho}(\partial\varphi)^2$, then we can write this Lagrange density cleanly as:

$$\boxed{\mathcal{L} = -2\left(e^{-2\phi} - \frac{\kappa}{2}(Z + w\phi)\right)(\partial^2\rho) - \frac{\kappa}{4}(\partial Z)^2 + 2e^{2\rho} + 2e^{-2\phi}(\partial\phi)^2 - 2\Lambda e^{2(\rho-\phi)}} \tag{4.23}$$

4.5.2 Z Field EoM

We are now ready to derive the EoM. The Lagrange density equation for the Z field as per 4.22 and Lagrangian 4.23 is:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial Z} = 2(\frac{\kappa}{2})(-\partial^2\rho), \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu Z)} = -\frac{\kappa}{2}\eta^{\mu\nu}\partial_\nu Z \implies \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu Z)}\right) = -\frac{k}{2}\partial^2 Z \\ \implies -\kappa\partial^2\rho + \frac{\kappa}{2}\partial^2 Z = 0 \\ \implies \partial^2 Z - 2\partial^2\rho = 0 \end{aligned} \tag{4.24}$$

Note that if we multiply this last expression by $e^{-2\rho}$ and use the fact that from 4.13: $\tilde{R} = -2e^{-2\rho}\partial^2\rho$ and $\Box = g^{\mu\nu}\partial_\mu\partial_\nu = e^{-2\rho}\partial^2$, then we can see that this equation can be written as:

$$\Box Z = -\tilde{R} \tag{4.25}$$

Which we used above in the introduction of the Z field.

4.5.3 ϕ Field EoM

For the dilaton field ϕ the EoM 4.22 from Lagrangian 4.23 are:

$$\frac{\partial \mathcal{L}}{\partial \phi} = 4e^{-2\phi}(\partial^2\rho) - (\partial^2\rho)\kappa w + 4e^{2(\rho-\phi)}\Lambda - 4e^{-2\phi}(\partial\phi)^2$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 4e^{-2\phi}\eta^{\mu\nu}\partial_\nu\phi \implies \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 8e^{-2\phi}(\partial\phi)^2 - 4e^{-2\phi}\partial^2\phi$$
$$\implies 4e^{-2\phi}(\partial^2\rho) - (\partial^2\rho)\kappa w + 4e^{2(\rho-\phi)}\Lambda - 4e^{-2\phi}(\partial\phi)^2 - 8e^{-2\phi}(\partial\phi)^2 + 4e^{-2\phi}\partial^2\phi = 0$$

Dividing through by $4e^{-2\phi}$ we have:

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\partial^2\rho - (\partial^2\phi) + (\partial\phi)^2 + e^{2\rho}\Lambda = 0 \tag{4.26}$$

4.5.4 ρ Field EoM

We need to be a little more careful when trying to find the equations of motion of the ρ field. If we look closely at the Lagrangian density 4.23 we can see that it contains $\partial^2\rho$ which in turn contains second derivatives of ρ. We usually do not expect to see such terms in our Lagrangian. This is mainly because it was shown in the 1800's by Mikhail Vasilyevich Ostrogradsky that any non-degenerate Lagrangian that is a function of 2nd time derivatives or higher will lead to an unstable Hamiltonian. Thankfully this is dependent on the Lagrangian being non-degenerate but the Lagrangian density 4.23 is degenerate (as there are no quadratic terms or higher in $\partial\rho$ or $\partial^2\rho$). This means this is a constrained system and indeed the fields must satisfy constraints which are written down in Bousso's and Hawking's paper [3].

While not unstable like the systems Ostrogradsky dealt with, we still have to treat our Lagrangian as Ostrogradsky did due to the presence of the $\partial^2\rho$ term. In appendix C.2.3 we show that a Lagrangian depending on such second order derivatives of a field results in the equations of motion C.4. Thus using C.4 yields the following:

$$\frac{\partial \mathcal{L}}{\partial \rho} = -4e^{2(\rho-\phi)}\Lambda + 4e^{2\rho}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\rho)} = 0, \quad \frac{\partial \mathcal{L}}{\partial(\partial^2\rho)} = -2\left(e^{-2\phi} + \frac{\kappa}{2}(Z + w\phi)\right)$$
$$\implies \partial^2\left(\frac{\partial \mathcal{L}}{\partial(\partial^2\rho)}\right) = 4e^{-2\phi}\partial^2\phi - 8e^{-2\phi}(\partial\phi)^2 - \kappa\partial^2 Z - \kappa w\partial^2\phi$$
$$\implies \left(4e^{-2\phi} - \kappa w\right)\partial^2\phi - 8e^{-2\phi}(\partial\phi)^2 - \kappa\partial^2 Z + 4e^{2\rho}\left(1 - \Lambda e^{-2\phi}\right) = 0$$

Dividing through by $-4e^{-2\phi}$ we are left with:

$$-\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\partial^2\phi + 2(\partial\phi)^2 + \frac{\kappa}{4}e^{2\phi}\partial^2 Z + 4e^{2(\rho+\phi)}\left(\Lambda e^{-2\phi} - 1\right) = 0 \tag{4.27}$$

4.5.5 Constraint Equations

As mentioned before there are equations of constraint associated with the Lagrangian 4.23. These come from varying the Lagrangian wrt to the metric elements g_{01} and g_{10}. i.e. these constraints make sure the metric stays diagonal. This variant results in the following constraints:

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\left(\delta^2\phi - 2\delta\phi\,\delta\rho\right) - (\delta\phi)^2 = \frac{\kappa}{8}e^{2\phi}\left[(\delta Z)^2 + 2\delta^2 Z - 4\delta Z\,\delta\rho\right]; \tag{4.28}$$
$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\left(\dot{\phi}' - \dot{\rho}\phi' - \rho'\dot{\phi}\right) - \dot{\phi}\phi' = \frac{\kappa}{8}e^{2\phi}\left[\dot{Z}Z' + 2\dot{Z}' - 2(\dot{\rho}Z' + \rho'\dot{Z})\right]. \tag{4.29}$$

4.6 Stability of Perturbations

We are now able to discuss the main purpose of Bousso and Hawking's paper, that is analyzing the stability of perturbations to the near maximal mass limit (i.e. near the Nariai solution). In 4.2.1 we showed that very close to the Nariai limit the geometry of the becomes that of the metric 4.9, comparing this to the ansatz we made in 4.4 we noticed that this gave us the relations 4.10 for the coefficients of our ansatz. We can now use these relations along with the previously calculated equations of motion 4.24, 4.26 and 4.27, to calculate how these relations change under the addition of quantum effects. Initially near the maximal solution, the radius of the two sphere $e^{-\phi}$ will remain constant, the value how-ever will no longer be $1/\sqrt{\Lambda}$, similarly the $1/\Lambda$ in the $e^{2\rho}$ term (Which is the radius of the one sphere squared), will be shifted. We can solve for the changes to these terms by the following ansatz:

$$e^{2\rho} = \frac{1}{\Lambda_1 \cos^2 t}, \quad e^{2\phi} = \Lambda_2 \tag{4.30}$$

The values these take will be governed by the equations of motion. Before we plug in it is useful to consider what limits we will be taking later so as to help us define some useful parameters. Bousso and Hawking, as we will see later, take the large N limit, where N is the number of different scalar fields introduced to our metric. This is so that the quantum fluctuations of metric are dominated by these fields. Large N means the parameter $\kappa = 2N/3$ is large $\kappa >> 1$. While it is desired for quantum effects to have notable change, we still want the changes to be small compared to the whole spacetime as we are still using a semi-classical approach that does not consider any theory of quantum gravity. Hence while κ is large, $b \equiv \kappa\Lambda$ will be small $b << 1$. It will be useful to write our quantities in terms of b where we can so that we may expand in this parameter later on.

4.6.1 Calculating Shifts in Λ

We can now proceed to sue the equations of motion 4.24, 4.26 and 4.27, to calculate the values of Λ_1 and Λ_2. We start by noticing that we can calculate $\partial^2\rho$ in the following way. using the ansatz for ρ:

$$\begin{aligned} &\partial_t\left(e^{2\rho}\right) = 2\dot{\rho}e^{2\rho} \implies \dot{\rho} = \frac{1}{2}e^{-2\rho}\partial_t\left(e^{2\rho}\right) = \frac{1}{2}\Lambda_1\cos^2 t\,\partial_t\left(\frac{1}{\Lambda_1\cos^2 t}\right) = \cos^2 t\cdot\frac{\tan t}{\cos^2 t} = \tan t \\ &\qquad \implies \ddot{\rho} = \frac{1}{\cos^2 t} = \Lambda_1 e^{2\rho} \end{aligned} \tag{4.31}$$

From the ansatz 4.30 we can see that $\rho = \rho(t)$, so $\partial^2\rho = -\ddot{\rho} + \rho'' = -\ddot{\rho} = -\Lambda_1 e^{2\rho}$.

The Z equation of motion 4.24, can be used to eliminate Z from the ρ equation of motion 4.27, by replacing $\partial^2 Z$ with $\partial^2\rho$. Since the in ansatz 4.30 ϕ is a constant, all the derivatives vanish so we can neglect the $(\partial\phi)^2$ and $\partial^2\phi$ terms. Using all of this the ρ equation of motion becomes:

$$\begin{aligned} &\frac{\kappa}{4}e^{2\phi}(2\partial^2\rho) + 4e^{2(\rho+\phi)}\left(\Lambda e^{-2\phi} - 1\right) = 0 \\ \implies &\frac{\kappa}{2}\partial^2\rho + e^{2\rho}\left(\frac{\Lambda}{\Lambda_2} - 1\right) = 0 \end{aligned}$$

Using the above derived expression for $\partial^2\rho = -\Lambda_1 e^{2\rho}$ we have:

$$\boxed{-\frac{\kappa}{2}\Lambda_1 + \frac{\Lambda}{\Lambda_2} - 1 = 0} \tag{4.32}$$

In a similar manner the ϕ equation of motion 4.26 can be written as:

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\partial^2\rho + e^{2\rho}\Lambda = 0$$
$$\implies -\left(1 - \frac{w\kappa}{4}\Lambda_2\right)\Lambda_1 e^{2\rho} + e^{2\rho}\Lambda = 0$$

So we have the second relation:

$$\boxed{-\left(1 - \frac{w\kappa}{4}\Lambda_2\right)\Lambda_1 + \Lambda = 0} \tag{4.33}$$

We now, with 4.32 and 4.33 have a system of two equations and two unknowns, which can be easily solved. Seeing as these equations are not linear but rather quadratic, there are two sets of solutions. However, only one of these corresponds to $1/\Lambda_1 > 0$ and $\Lambda_2 > 0$. The solutions are:

$$\frac{1}{\Lambda_1} = \frac{1}{8\Lambda}\left[4 - (w+2)b + \sqrt{16 - 8(w-2)b + (w+2)^2b^2}\right] \tag{4.34}$$

And:

$$\Lambda_2 = \frac{1}{2wk}\left[4 + (w+2)b - \sqrt{16 - 8(w-2)b + (w+2)^2b^2}\right] \tag{4.35}$$

If we expand these up to first order in b we find that:

$$\frac{1}{\Lambda_1} = \frac{1}{\Lambda}\left(1 - \frac{wb}{4}\right) + \mathcal{O}(b^2), \quad \Lambda_2 = \Lambda\left(1 - \frac{b}{2}\right) + \mathcal{O}(b^2)$$

So we see that in the presence of at least small quantum effects, both the radii of the one sphere and two sphere are smaller then the classical case of $\frac{1}{\sqrt{\Lambda}}$.

4.6.2 Perturbing Two Sphere Radius

The main stability Bousso and Hawking wish to analyze is the effect of perturbing the 2-sphere radius $e^{-\phi}$. Changing this radius changes the original 4 dimensional radius and in turn changes the radii of the two horizons as we will see later. We discussed at the end of subsection 4.2.1 that the co-ordinates χ and x are closely related as they are both co-ordinates along the one sphere and have the same values at both of the horizons. Thus for a small perturbation along the two sphere we can consider the change in this radius as a function of the compact co-ordinate x to be very similar as the change we see in equation 4.3. In this equation when the black hole is just below the maximal mass, the radius squared r^2 is, to first order in epsilon $r^2 = e^{-2\phi} = \frac{1}{\Lambda}(1 - 2\epsilon\cos\chi)$. Bousso and Hawking use this fact to make the following ansatz for the perturbation along the two sphere:

$$e^{2\phi} = \Lambda_2\left[1 + 2\epsilon\sigma(t)\cos x\right] \tag{4.36}$$

Where $\sigma(t)$ is called the *metric perturbation*. Notice also that this is $1/r^2$ and that we have expanded to first order in epsilon, hence the sign change compared to the term in 4.3. We could also make a similar perturbation of the radius of the one sphere e^{ρ}, but as it is pointed out by Bousso and Hawking, and as we will show later, this perturbation would not enter the equations of motion for σ at first order in ϵ.

Metric Perturbation

We can use our equations of motion for the fields to determine the form of this metric perturbation $\sigma(t)$. These will give us a differential equation that we can then solve by considering different scenarios. This differential equation is derived in appendix C.3. The calculation is performed by using the Z and ϕ equations of motion, 4.24 and 4.26, to write $\partial^2 Z$ in terms of $\partial^2\rho$ and then in turn $\partial^2\rho$ in terms of the field ϕ and other terms. The ρ equation of motion is then used to solve for the differential equation for leading order in ϵ. The result is as follows:

$$\boxed{\frac{\ddot{\sigma}}{\sigma} = \frac{a}{\cos^2 t} - 1} \tag{4.37}$$

Where:

$$a = \frac{2\sqrt{16 - 8(w-2)b + (w+2)^2 b^2}}{4 - wb} \tag{4.38}$$

Expanding this to first order in b we find that:

$$a = 2 + b + \mathcal{O}(b^2)$$

4.6.3 Horizon Tracing

The next step of the stability analysis is to locate the position of the black hole and cosmological horizons. It is a little non-trivial to locate the positions of the horizons once quantum effects have been introduced. In terms of our conformal co-ordinates x and t, we do not simply have a function $f(r)$ that vanishes at special points of the metric. Instead we can recall some other special properties of horizons. Horizons are the points where it becomes impossible to stay at a constant radius. Such an observer is called static and has their velocity vector u^μ proportional to the vector[10] ∂_t.

Horizon locations

With this consideration we can proceed to find the locations of the horizons in the conformal co-ordinates. We follow [10] in noticing that a sensible definition of the apparent horizon is where $(\nabla\phi)^2$ becomes null. This is because when this vector becomes space-like one is inevitable dragged to stronger and stronger coupling as in a classical black hole. Notice that in the action 4.21, e^ϕ takes on the role of the gravitational coupling as its inverse square appears in front of the gravitational terms.

[10] This is because they are stationary wrt to an observer at spatial infinity, who's co-ordinates are the exactly the t and r found in the Schwarzschild metric.

Horizon Perturbation

With this condition we can see using ϕ' and $\dot{\phi}$ from C.5 we can see that:

$$(\nabla\phi)^2 = e^{2\rho}\left[(\partial_x\phi)^2 - (\partial_t\phi)^2\right] = 0$$
$$\implies \frac{\epsilon^2\sigma^2\sin^2 x - \epsilon^2\dot{\sigma}^2\cos^2 x}{1+2\epsilon\sigma\cos x} = 0$$
$$\implies \left(\frac{\dot{\sigma}}{\sigma}\right)^2 = \tan^2 x$$

This only takes place for the values of x that correspond to the horizons, there are two solutions to this equation; $\frac{\dot{\sigma}}{\sigma} = \pm\tan x$. The positive solution has:

$$x_b(t) = \arctan\left|\frac{\dot{\sigma}}{\sigma}\right| \tag{4.39}$$

And will always correspond to the black hole horizon as $\arctan\left|\frac{\dot{\sigma}}{\sigma}\right| \leq \frac{\pi}{2}$. To make the other solution have $x \in [0,\pi]$[11] we can write $-\tan x$ as $\tan(\pi - x)$, then the x corresponding to the cosmological horizon is:

$$x_c(t) = \pi - x_b(t) = \pi - \arctan\left|\frac{\dot{\sigma}}{\sigma}\right|$$

With these values of x we can find the radii of the horizons by plugging into 4.36 since $r = e^{-\phi}$. This tells us that:

$$\frac{1}{r_b(t)^2} = e^{2\phi[t,x_b(t)]} = \Lambda_2\left[1+2\epsilon\delta(t)\right] \tag{4.40}$$

Where we have defined the *horizon perturbation* $\delta(t)$ as:

$$\delta(t) \equiv \sigma\cos x_b = \frac{\sigma}{\sqrt{1+\left(\frac{\dot{\sigma}}{\sigma}\right)^2}} \tag{4.41}$$

Where in the last step we have used 4.39 and the fact that $\cos\arctan\theta = (1+x)^{-1/2}$. 4.40 tells us the radius of the black hole; the radius of the cosmological horizon can be found by simply replacing $\delta(t)$ with $-\delta(t)$ as $\cos(\pi - \arctan\theta) = -\cos\arctan\theta = -(1+x)^{-1/2}$. This gives rise to a smaller $\frac{1}{r^2}$ and hence a larger radius as expected. We can also see from 4.40 and 4.41 that if the horizon perturbation grows then the two horizons grow further apart, corresponding to evaporation.

All that is left to do is to solve 4.37 for $\sigma(t)$ and plug it into the above equations to get the evolution of the horizons. Since 4.37 is a seconder order ODE, it will have two initial conditions that need to be fixed.

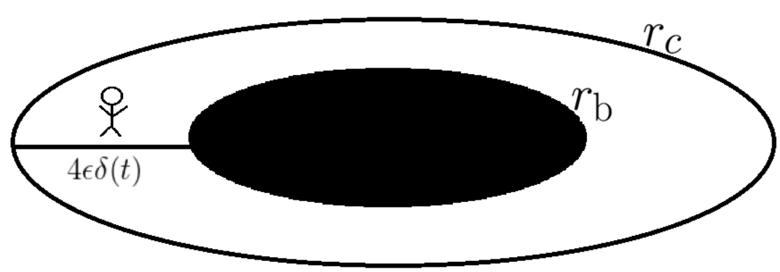


Figure 4.1: *Black hole and cosmological horizon separation*

[11] Recall that we require $x \in [0,\pi]$ as the horizons are initially located at $x = 0$ and $x = \pi$.

4.6.4 Classical Evolution

As a sanity check, Bousso and Hawking check the classical case with no scalar fields, meaning $\kappa = 0$ (ans so $b = 0$ also). This implies that $a = 2$, making it possible to solve the ODE 4.37. Since $\kappa = 0$ it is actually possible to use the constraint equations to simplify the differential equation 4.37. For $\kappa = 0$ 4.28 reads:

$$\delta^2\phi - 2\delta\phi\delta\rho - (\delta\phi)^2 = 0 \tag{4.42}$$

We can recall from C.6 that $(\delta\phi)^2 = \dot{\phi}^2 + \phi'^2 = \mathcal{O}(\epsilon^2)$ and $\delta^2\phi = \ddot{\phi} + \phi'' = \frac{2\epsilon\cos x[\ddot{\sigma}-\sigma]}{1+2\epsilon\sigma\cos x}$. From 4.31 we also have that $\dot{\rho} = \tan t$ and $\rho' = 0$, meaning $\delta\phi\delta\rho = \dot{\phi}\dot{\rho} = \frac{\epsilon\dot{\sigma}\cos x}{1+2\epsilon\sigma\cos x}\tan t$. This means 4.42 becomes:

$$\begin{aligned} &\frac{2\epsilon\cos x\,[\ddot{\sigma}-\sigma]}{1+2\epsilon\sigma\cos x} - 2\frac{\epsilon\dot{\sigma}\cos x}{1+2\epsilon\sigma\cos x}\tan t + \mathcal{O}(\epsilon^2) = 0 \\ &\implies \ddot{\sigma} - \sigma - 2\dot{\delta}\tan t = 0 \end{aligned}$$

From 4.37 we have, since $a = 2$: $\ddot{\sigma} = \sigma\left(a/\cos^2 t + 1\right)$. This means $\ddot{\sigma} - \sigma - 2\dot{\delta}\tan t = 0$ becomes:

$$\begin{aligned} \sigma\left(\frac{2}{\cos^2 t} + 2\right) - 2\dot{\sigma}\tan t = 0 \\ \implies 2\sigma\tan^2 t - 2\dot{\sigma}\tan t = 0 \\ \implies \dot{\sigma} = \sigma\tan t \end{aligned}$$

This equation fixes one of the initial conditions of the second order ODE 4.37, in that we must now have that $\sigma_0 = 0$. We can then directly solve this equation by integrating using the u-sub $u = \cos t$, where we find:

$$\sigma = \frac{\sigma_0}{\cos t}$$

We can then plug this into 4.41 to find very nicely that:

$$\delta(t) = \frac{\frac{\sigma_0}{\cos t}}{\sqrt{1+\tan^2 t}} = \sigma_0 = \text{const}$$

So in the presence of no quantum fields, no evaporation occurs and the horizons stay fixed in place. As would be expected.

4.6.5 Quantum Evolution

With the classical case well understood and consistent with our expectations we can proceed to examine the case where $\kappa > 0$ and the quantum fields are turned on. When $a > 0$ it is very difficult to obtain analytic expression. Instead Bousso and Hawking solve the sigma equation 4.37 in a power series in t to see what the early stages of evaporation look like. We will solve for σ and hence δ using a power series. It turns out that the leading, non trivial order is quadratic in t, so we do not need to expand further then that for these purposes. We can make the ansatz that σ takes the following form:

$$\sigma(t) = \sigma_0 + \dot{\sigma}_0 t + \frac{1}{2}\ddot{\sigma}_0 t^2$$

We can then plug this into 4.37, which to leading order fixes the value of the constant $\ddot{\sigma}_0$ as it reads $\ddot{\sigma}_0 = \sigma_0(a-1)$. With this fixed, σ can be plugged into 4.41 to find $\delta(t)$.

It is useful at this point to consider the possible different initial conditions. Since $\kappa > 0$, the constraint equations 4.28 and 4.29 no longer fix any of the initial conditions as the Z fields can now be involved. Thus we are left with two, totally separate initial conditions for σ fixed by the values of σ_0 and $\dot{\sigma}_0$. It is thus useful to consider the two "modes" of excitation, that is the solutions of $\sigma(t)$ where only one of σ_0 and $\dot{\sigma}_0$ is non vanishing. These are Dirichlet and Neumann boundary condition respectively.

Initial Perturbation

Let us first examine the same conditions we had for the classical case, i.e. $\sigma_0 > 0$ and $\dot{\sigma}_0 = 0$. With this consideration we can see that $\dot{\sigma}/\sigma = \frac{(a-1)t}{1+(a-1)t^2}$. Plugging this into 4.41 we get:

$$\begin{aligned}\delta(t) &= \frac{\sigma_0\left(1+\frac{1}{2}(a-1)t^2\right)}{\sqrt{1+\frac{(a-1)^2t^2}{(1+(a-1)t^2)^2}}} = \sigma_0\left(1+\frac{1}{2}(a-1)t^2\right)\left[1-\frac{1}{2}(a-1)^2t^2\right]+\mathcal{O}(t^4)\\ &= \sigma_0\left[1-\frac{1}{2}(a-1)(a-2)t^2\right]+\mathcal{O}(t^4)\\ &\approx \sigma_0\left[1-\frac{1}{2}bt^2\right]\end{aligned}$$

Where we have used the fact that $a = 2 + b + \mathcal{O}(b^2)$ and have ignored any terms of order b^2. This result is very surprising. What it tells us is that in this particular mode when $\sigma_0 > 0$ and $\dot{\sigma}_0 = 0$, the black hole horizon, when perturbed from the maximal mass, will *grow* instead of shrink. Thus it can be said that the black hole *Anti-evaporates*! The reason this is surprising is that as we calculated in Chapter 3, the temperature of a black hole in inversely proportional to its mass, so as the black hole shrinks, losing mass, we expect its temperature to increase and hence for it evaporate even faster. What we have shown here is the opposite, the black hole seems to mediate, at least initially through radiation with the cosmological horizon; returning back towards the maximal size. This would seem to imply that these type of black holes are stable. However, this is not the only mode and the other modes do in fact lead to instabilities.

Initial "Push"

The other possible type of initial perturbation is when $\dot{\sigma}_0 > 0$ and $\sigma_0 = 0$. This describes the scenario where the two horizons start coincided but are given a initial "push" that make them grow apart after some time. With this initial condition to leading order in t, $\sigma(t) = \dot{\sigma}_0 t$ so $\dot{\sigma}/\sigma = 1/t$. This means $\delta(t)$ from 4.41 becomes:

$$\delta(t) = \frac{\dot{\sigma}_0 t}{\sqrt{1+\frac{1}{t^2}}} = \dot{\sigma}_0 t^2 + \mathcal{O}(t^4)$$

This is very different to the previous case, once the black hole horizon is given this push, it begins to grow and will, at least initially, mean the continued evaporation of the black hole. This means this mode of initial conditions is unstable. If a system has at least one mode of conditions that lead it to being unstable we are forced to call that system unstable as there is always at least one mechanism for its collapse.

This analysis of these two modes of perturbation conclude the main results of Bousso and Hawking's paper [3]. In the last section there is also a discussion involving the no boundary condition, a cosmological model invented by Hawking along with James Hartle, that posits "before" the big bang time was just another spatial dimension that then diverged to become what it is today. In this final section Bousso Hawking analyse the Nariai limit in the context of this no boundary model. They do this as a Nariai black hole is most likely to form from a pair creation process subject to the no boundary condition. In studying this no boundary condition, they find that it selects a linear combination of the two types of perturbations that were discussed above. Hence, it is concluded that primordial black holes are unstable. Further discussion of this is beyond the scope of this work.

Conclusion

In conclusion, this project has explored and documented key aspects of quantum field theory in curved spacetime, with a focus on black hole dynamics. Starting with the Unruh effect, accelerating observers perceiving particle creation in a vacuum were examined, leading to the understanding that particle and vacuum states are observer-dependent. Building on this foundation, the work analysed Hawking radiation, demonstrating how black holes emit radiation due to quantum effects near their event horizons. Finally, the concept of black hole anti-evaporation in the Schwarzschild-de Sitter spacetime was investigated, where quantum corrections can lead to unexpected dynamics, such as the black hole's expansion after perturbations. These results underscore that even in the absence of a theory of quantum gravity, black holes and horizons in general can be studied using a variety of methods, each one enriching our knowledge and understanding of them and further paving the way for future advancements towards such a theory.

Appendix A
The Unruh Effect

A.1 Accelerating Observers

The goal here is to find the co-ordinates that best describe an constant accelerating observer. We work in $1+1$-dimensional Minkowski space to simplify calculations. If we have an observer at the origin in the frame S and a constant accelerating observer (constant acceleration κ in their own frame), whose frame is denoted S'. Then the four acceleration is defined as:

$$\alpha^\mu = \frac{d}{d\tau}u^\mu = \frac{d}{d\tau}\left(\gamma(1,\mathbf{v})^T\right).$$

From the definition of proper time we have $\frac{d}{d\tau} = \gamma\frac{d}{dt}$, so:

$$\begin{aligned}\alpha^\mu &= \gamma\left[\frac{d\gamma}{dt}(1,\mathbf{v})^T + \gamma(0,\frac{d\mathbf{v}}{dt})^T\right] = \gamma\left[\gamma^3(\mathbf{v}\cdot\mathbf{a})(1,\mathbf{v})^T + \gamma(0,\mathbf{a})^T\right]\\ &= \left(\gamma^4(\mathbf{v}\cdot\mathbf{a}), \gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v} + \gamma^2\mathbf{a}\right)^T\end{aligned}$$

Where we have used the fact that $d\gamma/dt = \gamma^3\mathbf{v}\cdot\mathbf{a}$. Here $\mathbf{a} \equiv \frac{d\mathbf{v}}{dt}$

Then the two 4-accelerations α^μ and $\alpha'^\mu = (0,\kappa)$ are related via a Lorentz Transformation which is a function of the relative velocity v:

$$\alpha'^\mu = \Lambda^\mu{}_\nu(\mathbf{v})\ \alpha^\nu$$

In 2-dimensions $\mathbf{a} \to a$ is a scalar, so:

$$\begin{aligned}\alpha'^\mu = \begin{pmatrix}0\\ \kappa\end{pmatrix} &= \begin{pmatrix}\gamma & -\gamma v\\ -\gamma v & \gamma\end{pmatrix}\begin{pmatrix}\gamma^4(va)\\ \gamma^4(va)v + \gamma^2 a\end{pmatrix}\\ \implies \kappa &= -\gamma^5 v^2 a + \gamma^5 v^2 a + \gamma^3 a = \gamma^3 a \end{aligned} \tag{A.1}$$

So the stationary observer sees the constant accelerating observer, accelerating with an acceleration of $\kappa\gamma^{-3}$. We can now use this to solve for the trajectories that the accelerating observer must be following. From A.1 we have:

$$\begin{aligned}\frac{dv}{dt} = \kappa\gamma^{-3} &\implies \gamma\frac{dv}{dt} = \frac{dv}{d\tau} = \kappa\gamma^{-2}\\ \implies \tau - \tau_0 &= \frac{1}{\kappa}\int\frac{dv}{(1-v^2)}, \quad (v = \tanh\psi)\\ &= \frac{1}{\kappa}\int d\psi = \frac{1}{\kappa}\psi = \frac{1}{\kappa}\text{arctan}\,(v)\end{aligned}$$

Where we have used the fact that $1 - v^2 = 1 - \frac{\sinh^2\psi}{\cosh^2\psi} = \frac{1}{\cosh^2\psi} = \frac{1}{\gamma^2}$ and $dv = \frac{1}{\cosh^2\psi}d\psi$. This means that:

$$\begin{aligned} v = \frac{dv}{dt} &= \tanh\left(\kappa(\tau - \tau_0)\right) \\ \implies \gamma\frac{dv}{dt} = \frac{dv}{d\tau} &= \sinh\left(\kappa(\tau - \tau_0)\right), \quad (\gamma^{-1} = \cosh\psi) \\ \implies x(\tau) &= x_0 + \frac{1}{\kappa}\cosh(\kappa(\tau - \tau_0)) \end{aligned}$$

This is the trajectory the particle follows with respect to its own proper time. We would also like to know how this proper time relates to the time of the stationary observer. Returning to A.1 we have that:

$$\begin{aligned} \frac{dv}{dt} &= \kappa\gamma^{-3}, \quad \text{(recall: } v = \tanh\psi) \\ \implies t - t_0 &= \frac{1}{\kappa}\int \frac{dv}{(1-v^2)^{3/2}} = \frac{1}{\kappa}\int \left(\frac{1}{\cosh^2\psi}\right)^{1-3/2} d\psi \\ \implies t - t_0 &= \frac{1}{\kappa}\int \cosh\psi d\psi = \frac{1}{\kappa}\sinh\left(\arctan\left(v\right)\right) \end{aligned}$$

But we know that $\tau - \tau_0 = \frac{1}{\kappa}\arctan(v)$ so we can write that:

$$t - t_0 = \frac{1}{\kappa}\sinh(\kappa(\tau - \tau_0))$$

These two equations describe how for how an observer with a constant acceleration κ, moves through spacetime wrt to a stationary observer with local time t and position x. We can write these cleanly by setting $\tau_0 = t_0 = x_0 = 0$ and relabeling the constant acceleration from $\kappa \to a$:

$$t(\tau) = \frac{1}{a}\sinh(a\tau) \tag{A.2}$$

$$x(\tau) = \frac{1}{a}\cosh(a\tau) \tag{A.3}$$

A.1.1 Rindler metric

We can calculate the induced metric in Minkowski space (t, x), due to the Rindler co-ordinates 2.1 and 2.2, by inserting them into the Minkowski metric:

$$\begin{aligned} ds^2 = -dt^2 + dx^2 &= -\left(\cosh(\alpha\eta)e^{\alpha\xi}d\eta + e^{a\xi}\sinh(\alpha\eta)d\xi\right)^2 + \left(\cosh(\alpha\eta)e^{\alpha\xi}d\xi + e^{a\xi}\sinh(\alpha\eta)d\eta\right)^2 \\ &= e^{2\alpha\xi}\left[\left(\cosh^2(\alpha\eta) - \sinh^2(\alpha\eta)\right)d\xi^2 - \left(\cosh^2(\alpha\eta) - \sinh^2(\alpha\eta)\right)d\eta\right] \end{aligned}$$

$$\implies ds^2 = e^{2\alpha\xi}\left[-d\eta^2 + d\xi^2\right]t \tag{A.4}$$

From this we can see that we consider η the time co-ordinate and ξ the space co-ordinate. We can use this line element to read off the components of the metric. In matrix form this is:

$$(g_{\mu\nu}) = \begin{pmatrix} -e^{2\alpha\xi} & 0 \\ 0 & e^{2\alpha\xi} \end{pmatrix} \implies (g^{\mu\nu}) = \begin{pmatrix} -e^{-2\alpha\xi} & 0 \\ 0 & e^{-2\alpha\xi} \end{pmatrix} \tag{A.5}$$

A.1.2 Time-like Killing Vectors

We have stated already that η is the "time" co-ordinate for the Rindler co-ordinates, now we check that it is indeed a time-like Killing vector. Let us first do this for Region I, i.e. using co-ordinates 2.1 and 2.2. Using the chain rule since $\eta = \eta(t, x)$ we can write[1]:

$$\partial_\eta = \frac{\partial x}{\partial \eta}\partial_x + \frac{\partial t}{\partial \eta}\partial_t = e^{a\xi}\sinh(a\eta)\partial_x + e^{a\xi}\cosh(a\eta)\partial_t = a\left(t\partial_x + x\partial_t\right)$$

It is easy to show that the same holds for the co-ordinates 2.4 and 2.5. So this expression holds for Region II.

To check whether ∂_η is a time-like Killing vector we have to look at its norm. This is independent of co-ordinate system but is most easily seen in regular Minkowski co-ords (t, x):

$$(\partial_\eta)^\mu(\partial_\eta)_\mu = g_{\mu\nu}(\partial_\eta)^\mu(\partial_\eta)^\nu = a^2(-x^2 + t^2) < 0 \quad \text{(In Region I \& II } |x| > t) \tag{A.6}$$

Thus since the norm is always negative in Region I & II, ∂_η is time-like. It then follows from the definition of a Killing vector as K^μ such that $\nabla_{(\mu}K_{\nu)} \equiv \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$, that ∂_η is also a Killing vector as we can write[2]:

$$\begin{aligned}\nabla_\mu(\partial_\eta)_\nu &= \nabla_\mu\left(g_{\nu\sigma}(\partial_\eta)^\sigma\right) = \partial_\mu\left(-ax\delta^t_\nu + at\delta^x_\nu\right)\\ &= -a\delta^t_\nu\delta^x_\mu + a\delta^x_\nu\delta^t_\mu = -\left[a\delta^t_\nu\delta^x_\mu - a\delta^x_\nu\delta^t_\mu\right] = -\nabla_\nu(\partial_\eta)_\mu\end{aligned}$$

So ∂_η is a Time-like *Killing Vector* in Region I & II .

A.1.3 Future Directed Vectors

There is now the small issue of future directedness. For a given manifold M all vectors $X, Y \in T_pM$ the tangent space of the manifold at every point p can be split up into two equivalence classes, given by $X \sim Y \iff g(X, Y) < 0$. We then call one of these classes *future directed* and the other *past directed*. It obviously makes sense to call ∂_t future directed, so we just have to check when $g(\partial_\eta, \partial_t) < 0$ to find out when ∂_η is future directed.

$$g(\partial_\eta, \partial_t) = g_{\mu\nu}(\partial_\eta)^\nu(\partial_t)^\mu = g_{\mu\nu}(\partial_\eta)^\nu(\delta^\mu_t) = (-1)(ax)(1) = -ax$$

So we can see that this is future directed in Region I where $x > 0$, but is past directed in Region II where $x < 0$. To fix this we have to use $\partial_{-\eta} = -\partial_\eta$ as this has the same norm as in A.6 and is still a Killing vector, but has $g(\partial_\eta, \partial_t) = ax < 0$ in Region II.

This means our time-like future directed Killing vectors are ∂_η in Region I and $\partial_{-\eta}$ in Region II.

A.1.4 Klein Gordon Inner Product

We can think of our expression of the scalar field ϕ in terms of the plane wave modes in 1.3 as the expansion of the scalar field over a basis. I.e. we have just performed a change of basis. But since this is just a Fourier transform into momentum space, we are familiar with how to calculate

[1]From this form we can actually recognize ∂_η as just a times the boost vector in the $+x$ direction, which is expected for an observer that constantly has to boost to accelerate.

[2]Note that this calculation was in Minkowski co-ordinates, where all Christoffel symbols are 0.

the form of $\phi(p)$ (which are essentially the creation and annihilation operators) in the momentum basis, we just have to perform an inverse Fourier transform. in this sense we can think of this "operation of inverse transform" as an inner product over the vector space of plane wave modes that gives us non-vector quantities, the $\phi(p)$.

However if we change to a different co-ordinate system and the scalar field ϕ is no longer expressible in terms of plane waves, we will need to develop a proper criteria for choosing modes and calculating creation and annihilation operators. One thing we certainly expect from this inner product is that it should be independent of time. So the question is can we construct a quadratic (inner product needs to be bi-linear) combination of two fields ϕ_1 and ϕ_2 that is time independent. Recall that as part of Noether's theorem, if there exists a conserved 4-current $\partial_\mu j^\mu = 0$ then since this takes the form of a continuity equation the integral of 0th component of the 4-current must be a conserved "Noether charge". So all we need to do is find a conserved current, or equivalently a continuity equation for the Klein Gordan scalar field.

If we start with what we know ϕ satisfies, the Klein Gordan equation:

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0$$

Now we want to get a quadratic combination, so lets multiply by the complex conjugate ϕ^* [3] and subtract the complex conjugate of the same :

$$\phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^* + \left(\phi^* \phi - \phi \phi^*\right) m^2 = 0$$

The second term will vanish as these are scalar fields. Notice that we can then pull out a derivative from the first term as the cross terms vanish:

$$\partial_\mu \left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) = 0$$

Which gives us a conserved current $J^\mu = \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*$. From this the conserved charge is:

$$\int j^0 d^3x = \int \left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*\right) d^3x$$

Which has been constructed to be conserved. Now we can notice that since $\phi \partial_0 \phi^* = (\phi^* \partial_0 \phi)^*$, this is a complex number minus its conjugate, which is always imaginary. So to make this quantity real we have to multiply by i. This is then naturally what we should choose to be our inner product. It is called the *Klein Gordon inner product*:

$$(\phi_1, \phi_2) = i \int \left(\phi_1^* \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1^*\right) d^3x \qquad \text{(A.7)}$$

It can be checked that if we take the time derivative of this equation it vanishes as long as both fields satisfy the KG equation.

A.1.5 Orthogonality of Rindler Modes

Here we check that the Rindler modes defined in 2.7 and 2.8 are properly orthogonal and normalized with respect to the Klein Gordan inner product A.7. In the $1+1$ dimensional case, the integral

[3] Even if we deal with real scalar fields the expansion over modes can be complex as it is for plane waves.

over space, with measure d^3x becomes just a single integral over the spatial Rindler co-ordinate ξ. So:

$$\begin{aligned}(g_k^{(1)}, g_{k'}^{(1)}) &= i\int d\xi \left[g_k^{(1)*}(\partial_\eta g_{k'}^{(1)}) - g_{k'}^{(1)}(\partial_\eta g_k^{(1)*})\right]\\ &= i\int \frac{d\xi}{4\pi\sqrt{\omega_k\omega_{k'}}}\left[e^{i\omega_k\eta - ik\xi}(-i\omega_{k'})e^{-i\omega_{k'}\eta + ik'\xi} - e^{-i\omega_{k'}\eta+ik'\xi}(i\omega_k)e^{i\omega_k\eta - ik\xi}\right]\\ &= \frac{\omega_k+\omega_{k'}}{4\pi\sqrt{\omega_k\omega_{k'}}}e^{i\eta(\omega_k-\omega_{k'})}\int d\xi e^{i\xi(k'-k)} = \frac{\omega_k+\omega_{k'}}{4\pi\sqrt{\omega_k\omega_{k'}}}e^{i\eta(\omega_k-\omega_{k'})}(2\pi\delta(k'-k))\\ &= \delta(k-k')\end{aligned}$$

In a similar manner one can prove $(g_k^{(2)}, g_{k'}^{(2)}) = \delta(k-k')$. The product $(g_k^{(1)}, g_{k'}^{(2)})$ is clearly vanishing in Regions I and II as at least one of the modes vanish in each region. It is questionable whether they vanish in Regions III and IV, but we will show later that we can construct them to vanish. This means we can collectively write:

$$\boxed{\begin{aligned}(g_k^{(1)}, g_{k'}^{(1)}) &= \delta(k-k')\\ (g_k^{(2)}, g_{k'}^{(2)}) &= \delta(k-k')\\ (g_k^{(1)}, g_{k'}^{(2)}) &= 0\end{aligned}} \tag{A.8}$$

A.1.6 Left and Right Rindler Modes

When we split the Rindler modes into right and left moving in 2.16 we actually solve our problem of covering Regions III and IV of spacetime. Here is how. We want to figure out what lines of constant phases for these modes look like as lines of positive phase represent the direction the waves are traveling in. Think of this as following a point on a wave as it moves, if you follow in the direction its traveling at the same speed as the wave the phase appears not to change. We saw that in 2.17 that right moving modes in Region I are functions of $x-t$, which means lines of constant phase of this mode have $x-t=$ const, i.e. these are 45° lines with positive slope. What this means is that in Region I right moving modes can only come from Region IV, meaning they must have $x+t<0$.

Similarly if we look at 2.19 we see that these modes which are part of Region II also have line of constant phase with $x-t=$ const and thus also are 45° lines with positive slope. How ever since these modes must be part of Region II, these modes can only occupy Regions I and III, meaning they have $x+t>0$.

This discussion shows that the two modes 2.17 and 2.19 cover all of Minkowski space and are non-overlapping. The exact same argument can be repeated for the left moving modes, in this case one would find that lines of constant phase have $x+t=$ const and that the two modes occupy the regions with $x-t>0$ and $x-t<0$, leading again to two non-overlapping modes that cover all of spacetime. This is that given a fixed k the modes $g_k^{(1,2)}$ and $g_{-k}^{(2,1)}$ are defined in separate, non overlapping regions, so we can write:

$$\boxed{(g_k^{(1)}, g_{-k}^{(2)}) = 0} \tag{A.9}$$

A.1.7 Normalization of $h_k^{(1,2)}$ Modes

Here we derive the normalization of the modes 2.27 and 2.28 according to the Klein Gordon inner product as defined in A.7. We know from the discussion of in section 2.4 that $h_k^{(1)}$ must take the form:

$$h_k^{(1)} = A(\omega_k)\left[g_R^{(1)}(k) + (-1)^{\frac{i\omega}{a}} g_L^{(2)*}(k)\right] \tag{A.10}$$

Where $A(\omega)$ is the normalization factor, which is a function of the frequency $\omega = \omega_k$. We can make use of the choice of logarithm 2.33 which gets rid of the ambiguity by setting $(-1)^{-\frac{i\omega}{a}} = e^{-\frac{\pi\omega}{a}}$, from 2.34. For this mode to be properly normalized we must have that $(h_k^{(1)}, h_{k'}^{(1)}) = \delta(k-k')$ as we had for the Rindler modes in A.1.5. To avoid calculating an integral we can simply use this expression A.10 along with the orthogonality of the Rindler modes in A.8. This means:

$$\begin{aligned}(h_k^{(1)}, h_{k'}^{(1)}) &= A(\omega_k)A^*(\omega_{k'})\left(g_R^{(1)}(k) + e^{-\frac{\pi\omega_k}{a}} g_L^{(2)*}(k)\ ,\ g_R^{(1)}(k') + e^{-\frac{\pi\omega_{k'}}{a}} g_L^{(2)*}(k')\right) \\ &= A(\omega_k)A^*(\omega_{k'})\left[\left(g_R^{(1)}(k), g_R^{(1)}(k')\right) + e^{-\frac{\pi(\omega_k+\omega_{k'})}{a}}\left(g_L^{(2*)}(k), g_L^{(2*)}(k')\right)\right]\end{aligned}$$

Where here we don't have any cross terms as $g_R^{(1)}(k)$ and $g_L^{(2)*}(k)$ do not overlap, as discussed in A.1.6. We can then go ahead and use the results of A.8 along with the property ?? of of the KG inner product to find:

$$\begin{aligned}&= A(\omega_k)A^*(\omega_{k'})\left[\delta(k-k') + e^{-\frac{\pi(\omega_k+\omega_{k'})}{a}}[-\delta(k-k')]\right] \\ &= |A(\omega_k)|^2\delta(k-k')\left[1 - e^{-\frac{2\pi\omega_k}{a}}\right]\end{aligned}$$

This must be equal to $\delta(k-k')$, so choosing $A(\omega)$ to be real and positive we find:

$$\boxed{A(\omega) = \frac{1}{\sqrt{1 - e^{-\frac{2\pi\omega}{a}}}}} \tag{A.11}$$

A.1.8 Complex Theorem

Here we discuss why the Theorem in 2.5.1 is true. We can think about this using the Fourier transform, let us consider some complex function $f(u)$, we can write down the Fourier transform as:

$$f(u) = \int_{-\infty}^{\infty} \tilde{f}(\lambda)e^{-i\lambda u}d\lambda$$

Then if $\tilde{f}(\lambda)$ is non-zero for any $\lambda < 0$ then in the lower half plane ($\mathrm{Im}(u) < 0$) we can split the exponential factor into $e^{-i\lambda u} = e^{\lambda \mathrm{Im}(u)}e^{\lambda \mathrm{Re}(u)} = e^{(-|\lambda|)(-|\mathrm{Im}(u)|)}e^{-|\lambda|\mathrm{Re}(u)} = e^{|\lambda||\mathrm{Im}(u)|}e^{-|\lambda|\mathrm{Re}(u)}$, So we can see that as $\mathrm{Im}(u) \to -\infty$ this factor diverges. Since this holds for any $\lambda < 0$, this condition is sufficient and we can say that $f(u)$ being a function of positive frequencies is equivalent to it being analytic (as it needs to behave well, be smooth ect) and bounded in the lower half complex plane.

Appendix B
Hawking Radiation

B.1 Black Holes

B.1.1 Null Geodesic integral

Here we compute the integral in 1.8:

$$t - t_0 = \int \frac{dr}{\left(1 - \frac{2M}{r}\right)} = \int \frac{rdr}{(r - 2M)}$$

To do this we can integrate by parts, using $u = r, \implies du = dr$ and $v = \ln(r - 2M) \implies dv = dr/(r - 2M)$. This means:

$$\begin{aligned} t - t_0 = uv - \int vdu &= r\ln(r - 2M) - \int \ln(r - 2M)\, dr \\ &= r\ln(r - 2M) - (r - 2M)\ln(r - 2M) + (r - 2M) = 2M\ln(r - 2M) + (r - 2M) \\ &= r + 2M\ln\left(\frac{r}{2M} - 1\right) + C \end{aligned}$$

$$\boxed{t - t_0 = r + 2M\ln\left(\frac{r}{2M} - 1\right)} \tag{B.1}$$

Where in the last step we have absorbed the constant into the definition of t_0.

B.1.2 Eddington-Finkelstein metric

Here we show that the Eddington-Finkelstein metric is indeed 1.13. We can start from the tortoise metric 1.11 and notice that $t = \frac{1}{2}(u + v)$ and $r^* = \frac{1}{2}(v - u)$, so we have:

$$\begin{aligned} dt^2 &= \frac{1}{4}\left[du^2 + dudv + dv^2\right] \\ dr^{*2} &= \frac{1}{4}\left[du^2 - 2dudv + dv^2\right] \\ &\implies -dt^2 + dr^{*2} = -dudv \end{aligned}$$

Plugging this into 1.11 gives us the metric 1.13. If we instead want v and r we can use the fact that $\frac{dr}{f(r)} = dr^* = \frac{1}{2}(dv - du) \implies du = dv - \frac{dr}{f(r)}$ so:

$$-f(r)dudv = -f(r)dv^2 + 2dvdr$$

This gives us 1.12

Christoffel Symbols

Here we calculate the Christoffel symbol Γ^0_{00} in the Eddington-Finkelstein co-ordinate system (u, v, θ, ϕ), as it is used in the calculation of 3.10. We can see that the metric 1.13 in matrix form is:

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & -\frac{1}{2}f(r) & 0 & 0 \\ -\frac{1}{2}f(r) & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \implies (g^{\mu\nu}) = \begin{pmatrix} 0 & -\frac{2}{f(r)} & 0 & 0 \\ -\frac{2}{f(r)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Using the definition of the Christoffel symbols as $\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)$, we can see that:

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2}g^{0\alpha}\left(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}\right) \\ &= g^{01}\partial_0 g_{10} = -\frac{2}{f(r)}\partial_u\left(\frac{-f(r)}{2}\right) = \frac{1}{f(r)}\partial_u f(r)\end{aligned}$$

To calculate $\partial_u f(r)$, we need to employ the partial derivative chain rule, Since we can write $u = v - 2r^*$, we can consider u a function of v and r^*, this way $\frac{\partial}{\partial u} = \frac{\partial v}{\partial u}\frac{\partial}{\partial v} + \frac{\partial r^*}{\partial u}\frac{\partial}{\partial r^*}$, since v is constant along this geodesic, we must have that $\partial_v r = 0 \implies \partial_v f(r) = 0$. We can also see that since $r^* = \frac{1}{2}(v-u) \implies \left(\frac{\partial r^*}{\partial u}\right)_v = -1/2$. So we can write:

$$\Gamma^0_{00} = \frac{1}{f(r)}\partial_u f(r) = \frac{1}{f(r)}\left(-\frac{1}{2}\partial_{r^*}f(r)\right) = -\frac{1}{2}\frac{f(r)}{f(r)}\partial_r f(r) = -\frac{1}{2}f'(r) \tag{B.2}$$

Where we have used the fact that $\partial_{r^*} = f(r)\partial_r$, from B.12.

B.1.3 Kruskal Metric

Here we calculate the Kruskal metric 1.15. To do this we start with the Eddington-Finkelstein metric 1.13, then using the defining relations 1.14 we find that $dU = -Udu/4M$ and $dV = Vdv/4M$, plugging these into 1.13 we find:

$$\begin{aligned}ds^2 &= \frac{16M^2}{UV}\left(1 - \frac{2M}{r}\right)dUdV + r^2 d\Omega^2 \\ &= -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2 d\Omega^2\end{aligned}$$

Where in the last step we have used, $UV = -e^{(v-u)/4M} = e^{r^*/2M} = e^{r/2M}\left(r/2M - 1\right)$
$= \frac{r}{2M}e^{r/2M}\left(1 - 2M/r\right)$.

B.1.4 Penrose Metric

Here we derive the metric for Penrose diagrams 1.17. To do this we start with the Kruskal metric 1.15, then using the relations 1.16 we can see that since $\frac{d}{dU}\left(\arctan(U)\right) = \frac{1}{1+U^2}$, then:

$$\begin{aligned}d\left(\arctan(U)\right) &= \frac{dU}{1+U^2} = \frac{dU}{1+\tan^2 u'} \\ &= \cos^2 u' dU\end{aligned}$$

Since the same holds for $v' = \arctan(V)$, we can plug these into the metric 1.15 to obtain 1.17.

B.1.5 EoM in Curved Space-Time

Here we want to vary the action S wrt to a scalar field φ, noting that the action, since we are in curved space-time, the Lagrange density $\mathcal{L}(\varphi, \nabla_\mu\varphi)$ is a function of the co-variant derivative of the scalar field, this means when we integrate by parts, where we previously had a four derivative in the flat space-time case, we will now have a covariant derivative. This means varying the actions such that $\delta S = 0$, implies:

$$\delta S = \int d^4x\delta\mathcal{L} = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\nabla_\mu\varphi\right) = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right)\right)\delta\varphi$$

$$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right) = 0} \tag{B.3}$$

Where we have integrated by parts the second term (Using the fact that $\delta\nabla_\mu\varphi = \delta\partial_\mu\varphi$ as φ is a scalar) and assumed the fields vanish at infinity.

B.1.6 d'Alembertian in Curved Space-Time

Here we show that the term that comes from variation of the part of the action quadratic in the covariant derivative of a scalar field φ, is equivalent to the extended d'Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. We will start with this term and expand, using the fact that since φ is a scalar field the action of the covariant derivative reduces to the partial derivative $\nabla_\nu\varphi = \partial_\nu\varphi$:

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = g^{\mu\nu}\nabla_\mu(\partial_\nu\varphi) = g^{\mu\nu}\left(\partial_\mu\partial_\nu\varphi - \Gamma^\sigma_{\mu\nu}\partial_\nu\varphi\right)$$

Where we have used the definition of the covariant derivative in the last step. The Christoffel symbols are defined as $\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu})$. This allows us to expand $\Box\varphi$ as:

$$\begin{aligned}\Box\varphi &= g^{\mu\nu}\partial_\mu\partial_\nu\varphi - \frac{1}{2}g^{\sigma\lambda}\left[g^{\mu\nu}\partial_\nu g_{\mu\lambda} + g^{\mu\nu}\partial_\mu g_{\lambda\nu} - g^{\mu\nu}\partial_\lambda g_{\mu\nu}\right]\partial_\sigma\varphi \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\varphi - g^{\sigma\lambda}g^{\mu\nu}\partial_\nu g_{\mu\lambda}\partial_\sigma\varphi + \frac{1}{2}g^{\sigma\lambda}g^{\mu\nu}\partial_\lambda g_{\mu\nu}\partial_\sigma\varphi\end{aligned} \tag{B.4}$$

Where we have relabeled the indices on the second and third terms. We can find forms for each of these terms through the following arguments. First we establish the fact that since the metric is its own inverse $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma = \text{const}$, then this means that $\partial_\lambda(g^{\mu\nu}g_{\nu\sigma}) = 0$, this then means:

$$(\partial_\lambda g^{\mu\nu})g_{\nu\sigma} = -g^{\mu\nu}(\partial_\lambda g_{\nu\sigma})$$

$$\implies \partial_\lambda g_{\nu\sigma}\delta^\nu_\rho = -(\partial_\lambda g^{\mu\nu})g_{\nu\sigma}g_{\mu\rho} \qquad \implies \partial_\lambda g^{\mu\nu}\delta^\rho_\nu = -(\partial_\lambda g_{\nu\sigma})g^{\mu\nu}g^{\sigma\rho}$$

$$\implies \partial_\lambda g_{\rho\sigma} = -(\partial_\lambda g^{\mu\nu})g_{\nu\sigma}g_{\mu\rho} \qquad \implies \partial_\lambda g^{\mu\rho} = -(\partial_\lambda g_{\nu\sigma})g^{\mu\nu}g^{\sigma\rho} \tag{B.5}$$

We know one of the terms will involve a $\partial_\lambda\sqrt{|g|}$, we need to figure out what this term looks like when expanded. We can see that since the metric always has a negative determinant, $\sqrt{|g|} = \sqrt{-g}$:

$$\partial_\lambda\sqrt{|g|} = \partial_\lambda\left[(-g^{-1})^{-\frac{1}{2}}\right] = -\frac{1}{2}\left[(-g^{-1})^{-\frac{3}{2}}\right]\partial_\lambda(-g^{-1}) \tag{B.6}$$

To calculate $\partial_\lambda(-g^{-1})$ we can use the fact that about matrices that $e^{\text{Tr}(A)} = \det(e^A)$ which is equivalent to $\text{Tr}(\ln(M)) = \ln(\det M)$, where $\ln M \equiv A$ and $M \equiv e^{\ln M}$. We can then calculate [1]:

$$\begin{aligned} \partial_\lambda \text{Tr}(\ln M) &= \text{Tr}(\partial_\lambda(M)M^{-1}) \\ \partial_\lambda \ln(\det(M)) &= \frac{\partial_\lambda(\det(M))}{\det(M)} \\ \implies \text{Tr}(M^{-1}\partial_\lambda(M)) &= \frac{\partial_\lambda(\det(M))}{\det(M)} \end{aligned} \tag{B.7}$$

We can then let $M = g^{\mu\nu} \implies \det(M) = \frac{1}{g}$. From B.7 we then have that $\text{Tr}(g_{\mu\nu}\partial_\lambda(g^{\mu\nu})) = \frac{\partial_\lambda(g^{-1})}{g^{-1}}$ so we have:

$$\partial_\lambda(-g^{-1}) = -\frac{1}{g}g_{\mu\nu}\partial_\lambda(g^{\mu\nu}) = \frac{1}{g}g_{\mu\nu}\partial_\lambda(g_{\delta\gamma})g^{\mu\delta}g^{\gamma\nu} = \frac{1}{2}g^{\gamma\delta}(\partial_\lambda g_{\delta\gamma}) = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\mu\nu})$$

Where we have dropped the trace as the repeated indices imply it (only because $g_{\mu\nu}$ is diagonal) and the second step here makes use of the relation B.5. Plugging this into B.6:

$$\partial_\lambda\sqrt{|g|} = -\frac{1}{2}\left[(-g)^{\frac{3}{2}}\right](\frac{1}{g})g^{\mu\nu}\partial_\lambda(g_{\mu\nu}) = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\lambda g_{\mu\nu}$$

From this we can recognize the third term in B.4 is:

$$\frac{1}{2}g^{\sigma\lambda}g^{\mu\nu}\partial_\lambda g_{\mu\nu}\partial_\sigma\varphi = \frac{1}{\sqrt{|g|}}g^{\sigma\lambda}\partial_\lambda\sqrt{|g|}\partial_\sigma\varphi = \frac{1}{\sqrt{|g|}}g^{\mu\nu}\partial_\mu\sqrt{|g|}\partial_\nu\varphi \tag{B.8}$$

For the second term in B.4 we can use B.5 to write it as:

$$g^{\sigma\lambda}g^{\mu\nu}\partial_\nu g_{\mu\lambda}\partial_\sigma\varphi = -g^{\sigma\lambda}g^{\mu\nu}\partial_\nu g^{\delta\gamma}g_{\delta\mu}g_{\gamma\lambda}\partial_\sigma\varphi = -\delta^\nu_\delta\partial_\nu g^{\delta\gamma}\delta^\sigma_\gamma\partial_\sigma\varphi = -\partial_\nu g^{\nu\gamma}\partial_\gamma\varphi = -\partial_\mu g^{\mu\nu}\partial_\nu\varphi \tag{B.9}$$

This means plugging B.8 and B.9 into B.4 we get:

$$\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = g^{\mu\nu}\partial_\mu\partial_\nu\varphi + \partial_\mu g^{\mu\nu}\partial_\nu\varphi + \frac{1}{\sqrt{|g|}}g^{\mu\nu}\partial_\mu\sqrt{|g|}\partial_\nu\varphi$$

$$\boxed{\implies \Box\varphi = \frac{1}{\sqrt{|g|}}\partial_\mu\left[\sqrt{|g|}g^{\mu\nu}\partial_\nu\varphi\right]} \tag{B.10}$$

As needed.

[1]To see why the first calculation here holds, consider $dM/M = d\left(\sum_{n=0}^{\infty}(\ln M)^n/n!\right)/M = d(\ln(M))$

B.2 Wave Equation

B.2.1 Tortoise Wave equation

Here we derive the wave equation with the Tortoise co-ordinate r^*. We show that it reduces to the wave equation with some potential $V(r)$. If we divide 3.5 by $\frac{1}{r}Y_{\ell m}$ and use 3.6, we can write:

$$\begin{aligned}
0 &= -\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2 f + \frac{1}{r}\partial_r\left(\left(r^2-2Mr\right)\left[\frac{\partial_r f}{r}-\frac{f}{r^2}\right]\right) - \frac{\ell(\ell+1)}{r^2}f \\
&= -\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2 f + \frac{2}{r^2}(r-M)\left[\partial_r f - \frac{f}{r}\right] + (r-2M)\left[\frac{\partial_r^2 f}{r} - \frac{\partial_r f}{r^2} - \frac{\partial_r f}{r^2} + 2\frac{f}{r^3}\right] - \frac{\ell(\ell+1)}{r^2}f \\
&= -\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2 f + \left(1-\frac{2M}{r}\right)\partial_r^2 f + \frac{2M}{r^2}\left[\partial_r f - \frac{f}{r}\right] - \frac{\ell(\ell+1)}{r^2}f = 0 \qquad \text{(B.11)}
\end{aligned}$$

We can then change the radial co-ordinate r to the tortoise co-ordinate r^* defined in 1.10. With this the partial derivative change via:

$$\begin{aligned}
\partial_r &= \frac{dr^*}{dr}\partial_{r^*} = \left(1-\frac{2M}{r}\right)^{-1}\partial_{r^*} \\
\partial_r^2 &= \partial_r\left(\left(1-\frac{2M}{r}\right)^{-1}\partial_{r^*}\right) \qquad \text{(B.12)} \\
&= -\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{2M}{r^2}\right)\partial_{r^*} + \left(1-\frac{2M}{r}\right)^{-2}\partial_{r^*}^2
\end{aligned}$$

With this we can see the middle two terms of B.11 become:

$$\begin{aligned}
&\left(1-\frac{2M}{r}\right)\partial_r^2 f + \frac{2M}{r^2}\left[\partial_r f - \frac{f}{r}\right] = \\
&-\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{2M}{r^2}\right)\partial_{r^*} f + \left(1-\frac{2M}{r}\right)^{-1}\partial_{r^*}^2 + \frac{2M}{r^2}\left[\left(1-\frac{2M}{r}\right)^{-1}\partial_{r^*} f - \frac{f}{r}\right] \\
&= \left(1-\frac{2M}{r}\right)^{-1}\partial_{r^*}^2 - \frac{2M}{r^3}f
\end{aligned}$$

So we can write B.11 as:

$$-\partial_t^2 f + \partial_{r^*}^2 f - \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)f = 0$$

$$\boxed{\left[-\partial_t^2 + \partial_{r^*}^2 - V(r)\right]f = 0} \qquad \text{(B.13)}$$

As needed.

B.3 Bogolubov Transformations

Often we can expand a scalar field ϕ over two sets of basis modes, f_k and g_k:

$$\begin{aligned}\phi &= \int_{-\infty}^{\infty} dk \left[a_k f_k + a_k^\dagger f_k^*\right] \\ &= \int_{-\infty}^{\infty} dk' \left[b_{k'} g_{k'} + b_{k'}^\dagger g_{k'}^*\right]\end{aligned}$$

Then if both these basis modes satisfy the same form of the $\Box\phi = 0$ equation, and are themselves a complete orthonormal set, then it is always possible to express one mode in terms of the other:

$$g_k = \int_{-\infty}^{\infty} dk' \left[\alpha_{kk'} f_{k'} + \beta_{kk'} f_{k'}^*\right] \tag{B.14}$$

Here $\alpha_{kk'}$ and $\beta_{kk'}$ are components of matrices. Since we are in the continuum limit these are infinite dimensional matrices. Their interpretation as matrices makes more sense in the discrete case.

As mentioned these two sets of modes f_k and g_k are orthonormal sets, meaning they satisfy the following relations wrt to the KG inner product A.7:

$$\begin{aligned}(f_k, f_{k'}) &= \delta(k-k'), \quad (g_k, g_{k'}) = \delta(k-k') \\ (f_k, f_{k'}^*) &= 0, \qquad\qquad (g_k^*, g_{k'}) = 0\end{aligned} \tag{B.15}$$

With these we can immediately see that:

$$\begin{aligned}(g_k, f_{k'}) &= \alpha_{kk'}, \quad (g_k, f_{k'}^*) = -\beta_{kk'} \\ (f_k, g_{k'}) &= \alpha_{k'k}^*, \quad (f_k, g_{k'}^*) = \beta_{k'k}\end{aligned} \tag{B.16}$$

Where the minus on the top right term appears due to the fact that: $(f_k^*, f_{k'}^*) = -(f_k, f_{k'}) = -\delta(k-k')$, as per ??.

The bottom two expressions follow from the top two in the following way. For any two modes, as per ??, we have $(\phi_1, \phi_2)^* = (\phi_2, \phi_1) \implies (f_k, g_{k'}) = (g_{k'}, f_k)^* = \alpha_{k'k}^*$. Also per ??, $(\phi_1, \phi_2) = -(\phi_1^*, \phi_2^*) \implies (f_k, g_{k'}^*) = -(f_k^*, g_{k'})^* = -(g_{k'}, f_k^*) = \beta_{k'k}$. This means the corresponding expansion of f_k in terms of $g_{k'}$ is:

$$f_k = \int_{-\infty}^{\infty} dk' \left[\alpha_{k'k}^* g_{k'} - \beta_{k'k} g_{k'}^*\right] \tag{B.17}$$

These are *Bogolubov Transformations.*

B.3.1 Completeness Relations

Here we prove some useful properties that come from the above Bogolubov transformations B.14 and B.17. If we insert B.14 into the relation $(g_k, g_{k'}) = \delta(k-k')$:

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} dp \left[\alpha_{kp} f_p + \beta_{kp} f_p^*\right], \int_{-\infty}^{\infty} dp' \left[\alpha_{k'p'} f_{p'} + \beta_{k'p'} f_{p'}^*\right]\right) = \delta(k-k') \\ \implies &\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \left[\alpha_{kp}\alpha_{k'p'}^* (f_p, f_{p'}) - \beta_{kp}\beta_{k'p'}^* (f_p^*, f_{p'}^*)\right] = \delta(k-k')\end{aligned}$$

Then using the orthogonality relations B.15:

$$\implies \int_{-\infty}^{\infty} dp \left[\alpha_{kp}\alpha^*_{k'p} - \beta_{kp}\beta^*_{k'p}\right] = \delta(k-k') \tag{B.18}$$

B.3.2 Operator Relations

From the expansions B.14 and B.17 we can retrieve expressions for relations between the two creation and annihilation operator sets a_k and b_k. If we take the inner product of (ϕ, g_k) and use the expansion of ϕ over f_k:

$$\begin{aligned}
(\phi, g_k) &= \left(\int_{-\infty}^{\infty} dk' \left[a_{k'} f_{k'} + a^\dagger_{k'} f^*_{k'}\right], g_k\right) \\
&= \int_{-\infty}^{\infty} dk' \left[a_{k'}(f_{k'}, g_k) + a^\dagger_{k'}(f^*_{k'}, g_k)\right] \\
&= \int_{-\infty}^{\infty} dk' \left[\alpha^*_{kk'} a_{k'} - \beta^*_{kk'} a^\dagger_{k'}\right]
\end{aligned}$$

Where we have used the relations in B.16. But we could also do this with the expansion of ϕ in terms of g_k:

$$(\phi, g_k) = \left(\int_{-\infty}^{\infty} dk' \left[b_{k'} g_{k'} + b^\dagger_{k'} g^*_{k'}\right], g_k\right) = b_k$$

Putting these together we have a relation between the two sets of creation and annihilation operators:

$$b_k = \int_{-\infty}^{\infty} dk' \left[\alpha^*_{kk'} a_{k'} - \beta^*_{kk'} a^\dagger_{k'}\right] \tag{B.19}$$

B.3.3 Number Operator

If at least one of f_k or g_k correspond to frames that are not inertial with respect to each other then we cannot expect these frames to have the same notion of the vacuum. Lets call the vacuum in the frame described by the modes f_k, $|0_f\rangle$. This means that the corresponding creation and annihilation operators obey: $a_k |0_f\rangle = 0$ and $\langle 0_f| a^\dagger_k = 0$. If the two frames do not have the same notion of the vacuum we cannot expect that $b_k |0_f\rangle = 0$. This means that if we calculate the expectation value of the number operator in the frame of reference of the g_k modes, $\langle N^{(g)}_k\rangle = \langle 0_f| g^\dagger_k g_k |0_f\rangle$, we will get a non zero result. Since the expectation value of the number operator corresponds to the number of particles in the state, this means that the observers in the frame with the g_k modes, will see the vacuum of the f_k modes to have a spectrum of particles.

We can see how the spectrum of particles depends on the quantities we calculated earlier, using our expression for b_k in terms of the a_k in B.19:

$$\begin{aligned}\langle N_k^{(g)}\rangle &= \langle 0_f| g_k^\dagger g_k |0_f\rangle \\ &= \langle 0_f| \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dp \left[\alpha_{kk'} a_{k'}^\dagger - \beta_{kk'} a_{k'}\right] \left[\alpha_{kp}^* a_p - \beta_{kp}^* a_p^\dagger\right] |0_f\rangle \\ &= \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dp \langle 0_f| \left[\beta_{kk'} \beta_{kp}^* a_{k'} a_p^\dagger\right] |0_f\rangle\end{aligned}$$

Where we have used the fact that $a_k |0_f\rangle = 0$ and $\langle 0_f| a_k^\dagger = 0$. We can then use the fact that a_k and $a_k^\dagger$ must obey the commutation relations $[a_k, a_{k'}^\dagger] = \delta(k - k') \implies \langle 0_f| a_{k'} a_p^\dagger |0_f\rangle = \langle 0_f| \delta(k' - p) + a_p^\dagger a_p |0_f\rangle = \delta(k' - p) \langle 0_f| |0_f\rangle = \delta(k' - p)$. This means that:

$$\langle N_k^{(g)}\rangle = \int_{-\infty}^{\infty} dp \beta_{kp} \beta_{kp}^* = \int_{-\infty}^{\infty} dp \beta_{kp} \beta_{pk}^\dagger$$

$$\boxed{\implies \langle N_k^{(g)}\rangle = \int_{-\infty}^{\infty} dp |\beta_{kp}|^2} \tag{B.20}$$

B.4 Killing Vectors

We want to be able to quantify how tensors change at different points in space. Unfortunately we cannot directly compare tensors at different points, so we have to take a different approach. If we instead think not of passive transformations but of active transformations, whereby we keep the points fixed and instead continuously change the co-ordinate system. The resulting derivative takes the form of the *Lie Derivative*.

B.4.1 Lie Derivative

This continuous change of the co-ordinate system can be carried out by defining the so-called *one parameter subgroups*. A one parameter subgroup of diffeomorphism $F_t(x)$ with associated vector field $\xi(x)$ is defined to *act on smooth tensors* $T = (T^{i_1,..,i_p}_{j_1,...,j_q})$ of type (p, q) as follows:

$$(F_t T)^{i_1,..,i_p}_{j_1,...,j_q}(x) = T^{k_1,..,k_p}_{l_1,...,l_q}(y) \frac{\partial y^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{l_1}}{\partial x^{j_q}} \frac{\partial x^{i_1}}{\partial y^{k_1}} \cdots \frac{\partial x^{i_p}}{\partial y^{k_p}}$$

Where $y^i = F_t^i(x) = \exp(t\xi^i)$.

The Lie derivative of a tensor $T = (T^{i_1,..,i_p}_{j_1,...,j_q})$ along a vector field ξ, can then be defined as the tensor $\mathcal{L}_\xi T$ given by:

$$\mathcal{L}_\xi T^{i_1,..,i_p}_{j_1,...,j_q} = \left[\frac{d}{dt}(F_t T)^{i_1,..,i_p}_{j_1,...,j_q}\right]_{t=0}$$

We can define an explicit formula for $\mathcal{L}_\xi$ by expanding $y^i = F_t^i(x) \approx x^i + t\xi^i(x)$ up to leading order and carrying out the derivative. This results in:

$$\begin{aligned}\mathcal{L}_\xi T^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \xi^a \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^a} + T^{a i_2 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial \xi^{i_1}}{\partial x^a} + \cdots + T^{i_1 \ldots i_{p-1} a}_{j_1 \ldots j_q} \frac{\partial \xi^{i_p}}{\partial x^a} \\ - T^{i_1 \ldots i_p}_{a j_2 \ldots j_q} \frac{\partial \xi^a}{\partial x^{j_1}} - T^{i_1 \ldots i_p}_{j_1 a j_3 \ldots j_q} \frac{\partial \xi^a}{\partial x^{j_2}} - \cdots - T^{i_1 \ldots i_p}_{j_1 \ldots j_{q-1} a} \frac{\partial \xi^a}{\partial x^{j_q}}\end{aligned}$$

For 2−Tensors, like the metric $g_{\mu\nu}$ this reduces to:

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu \xi^\sigma + g_{\mu\sigma}\partial_\nu \xi^\sigma \tag{B.21}$$

We can then transport this into an expression of co-variant derivatives, using metric comparability, which means $\nabla_\sigma g_{\mu\nu} = 0$. This way from the definition of the co-variant derivative we have:

$$\begin{aligned}
\nabla_\sigma g_{\mu\nu} &= \partial_\sigma g_{\mu\nu} - \Gamma^\rho_{\sigma\nu} g_{\rho\nu} - \Gamma^\rho_{\mu\sigma} g_{\mu\rho} = 0 \\
\nabla_\mu \xi^\sigma &= \partial_\mu \xi^\sigma + \Gamma^\sigma_{\mu\rho}\xi^\rho \\
\nabla_\nu \xi^\sigma &= \partial_\nu \xi^\sigma + \Gamma^\sigma_{\nu\rho}\xi^\rho
\end{aligned}$$

So re-arranging these expressions for the 4−derivative terms and plugging into B.21. we have:

$$\begin{aligned}
\mathcal{L}_\xi g_{\mu\nu} &= \xi^\sigma \Gamma^\rho_{\sigma\nu} g_{\rho\nu} + \xi^\sigma \Gamma^\rho_{\mu\sigma} g_{\mu\rho} + g_{\sigma\nu}\nabla_\mu \xi^\sigma - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho}\xi^\rho + g_{\mu\sigma}\nabla_\nu\xi^\sigma - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\xi^\rho \\
&= g_{\sigma\nu}\nabla_\mu\xi^\sigma + g_{\mu\sigma}\nabla_\nu\xi^\sigma \\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\end{aligned} \tag{B.22}$$

Where we have relabeled indices to cancel terms.

B.4.2 Killing Vector Definition

If we investigate the value of the metric along some curve generated by a vector ξ and the resulting values are the is the exact same, then intuitively we understand that there is a symmetry associated with that curve. This can easily be quantified by the Lie derivative along this curve being zero: $\mathcal{L}_\xi g_{\mu\nu} = 0$. We say that the vector generating this curve is called a *Killing Vector* [2]. We can then see from B.22 that the condition for the generating vector ξ to be a Killing vector is:

$$\boxed{\mathcal{L}_\xi g_{\mu\nu} = 0 \implies \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0} \tag{B.23}$$

This is often called the *Killing Equation*.

B.4.3 Conserved Quantities

From our familiarity with Noether's Theorem we expect symmetries to have associated conserved quantities. The same is true for Killing vectors. If $x^\mu(\lambda)$ is a geodesic with a tangent vector $U^\mu = \frac{dx^\mu}{d\lambda}$ and the metric has a killing vector ξ^μ. Then we can see that if we transport the quantity $\xi_\mu U^\mu = \xi_\mu \frac{dx^\mu}{d\lambda}$ is conserved along the geodesic:

$$\nabla_U (\xi_\mu U^\mu) = U^\nu \nabla_\nu (\xi_\mu U^\mu) = U^\nu (\nabla_\nu \xi_\mu) U^\mu + U^\nu \xi_\mu (\nabla_\nu U^\mu)$$

We can then recall that geodesics satisfy, $U^\nu \nabla_\nu U^\mu = 0$, so the second term here is 0. Then for the first term we can re-write it as:

$$U^\nu (\nabla_\nu \xi_\mu) U^\mu = \frac{1}{2} U^\mu U^\nu (\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu) = 0$$

Which we can recognize is 0, as ξ is a Killing vector. So $\xi_\mu \frac{dx^\mu}{d\lambda}$ is indeed a conserved quantity along this geodesic.

[2] Named after Wilhelm Killing

B.4.4 Killing Vectors from the Metric

If the metric in a certain co-ordinate system is independent of one of the co-ords x^α, then the vector $K = \partial_\alpha$ is a killing vector. We will show why this is true now. The components of K must be $K^\mu = \delta^\mu_\alpha$, $K_\mu = g_{\mu\nu}K^\nu = g_{\mu\alpha^*}$, where we have an asterisk on the α so that we do not confuse it with a regular index. We can then plug this into the killing equation B.23 to see if the result vanishes:

$$\begin{aligned}
\mathcal{L}_K g_{\mu\nu} &= \nabla_\mu K_\nu + \nabla_\nu K_\mu = \partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\sigma_{\mu\nu} K_\sigma \\
&= \partial_\mu g_{\nu\alpha^*} + \partial_\nu g_{\mu\alpha^*} - 2\Gamma^\sigma_{\mu\nu} g_{\sigma\alpha^*} \\
&= \partial_\mu g_{\nu\alpha^*} + \partial_\nu g_{\mu\alpha^*} - g^{\sigma\lambda} g_{\sigma\alpha^*} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right), \qquad (g^{\sigma\lambda} g_{\sigma\alpha^*} = \delta^\lambda_{\alpha^*}) \\
&= \partial_\mu g_{\nu\alpha^*} + \partial_\nu g_{\mu\alpha^*} - \partial_\mu g_{\alpha^*\nu} - \partial_\nu g_{\alpha^*\mu} + \partial_{\alpha^*} g_{\mu\nu} \\
&= \partial_{\alpha^*} g_{\mu\nu} = 0
\end{aligned}$$

Where we have used the fact that the metric is independent of the co-ordinate α so $\partial_{\alpha^*} g_{\mu\nu} = 0$. This shows that $K = \partial_\alpha$ is indeed a Killing vector. The converse of this statement is also true, if there exists a killing vector then there must exist a co-ordinate system in which the metric is independent of one of the co-ords.

Appendix C
Anti-Evaporation

C.1 Near Maximal Mass Metric

Here we show how the near maximal mass metric 4.3 comes from the Schwartzchild de Sitter metric 4.1. We start by expanding $f(r) = 1 - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2$ up to cubic order in ϵ. We can plug in the $r(\chi)$ co-ordinate from 4.2 as well as $\mu = \sqrt{\frac{1-3\epsilon^2}{9\Lambda}}$ to see that:

$$\begin{aligned} f(r) &= 1 - 2\sqrt{\frac{1-3\epsilon^2}{9\Lambda}}\frac{\sqrt{\Lambda}}{1-\epsilon\cos\chi - \frac{1}{6}\epsilon^2} - \frac{\Lambda}{3\Lambda}\left[1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right]^2 \\ &= 1 - \frac{2}{3}\frac{\sqrt{1-3\epsilon^2}}{1-\epsilon\cos\chi - \frac{1}{6}\epsilon^2} - \frac{1}{3}\left[1 - 2\epsilon\cos\chi - \frac{2}{6}\epsilon^2 + \epsilon^2\cos^2\chi + \frac{2}{6}\epsilon^3\cos\chi\right] + \mathcal{O}(\epsilon^4) \end{aligned} \quad \text{(C.1)}$$

For the fraction in the second term we can notice that it takes the form of a geometric sum:

$$\begin{aligned} \frac{1}{1-\epsilon\cos\chi - \frac{1}{6}\epsilon^2} &= \frac{1}{1-(\epsilon\cos\chi + \frac{1}{6}\epsilon^2)} = \sum_{n=0}^{\infty}(\epsilon\cos\chi + \frac{1}{6}\epsilon^2)^n \\ &= 1 + (\epsilon\cos\chi + \frac{1}{6}\epsilon^2) + (\epsilon\cos\chi + \frac{1}{6}\epsilon^2)^2 + \epsilon^3\cos^3\chi + \mathcal{O}(\epsilon^4) \\ &= 1 + \epsilon\cos\chi + \left(\frac{1}{6} + \cos^2\chi\right)\epsilon^2 + \cos\chi(\frac{1}{3} + \cos^3\chi)\epsilon^3 + \mathcal{O}(\epsilon^4) \end{aligned}$$

We can then plug this into the expansion C.1 along with the fact that $\sqrt{1-3\epsilon^2} = 1 - \frac{3}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)$, to get:

$$\begin{aligned} f(r) =& 1 - \frac{2}{3}\left[1 - \frac{3}{2}\epsilon^2\right]\left[1 + \epsilon\cos\chi + \left(\frac{1}{6} + \cos^2\chi\right)\epsilon^2 + \cos\chi(\frac{1}{3} + \cos^3\chi)\epsilon^3\right] \\ &- \frac{1}{3}\left[1 - 2\epsilon\cos\chi + (\cos^2\chi - \frac{1}{3})\epsilon^2 + \frac{1}{3}\cos\chi\epsilon^3\right] + \mathcal{O}(\epsilon^4) \\ =& 1 - \frac{2}{3}\left[\frac{3}{2} + (0)\epsilon + (\frac{3}{2}\cos^2\chi - \frac{3}{2})\epsilon^2 + (-\frac{3}{2}\cos\chi + \cos^3\chi + \frac{1}{3}\cos\chi + \frac{1}{6}\cos\chi)\epsilon^3\right] + \mathcal{O}(\epsilon^4) \\ =& \sin^2\chi\epsilon^2 - \frac{2}{3}\cos\chi(\cos^2\chi - 1)\epsilon^3 + \mathcal{O}(\epsilon^4) \end{aligned}$$

$$\boxed{\implies f(r) = \sin^2\chi(1 + \frac{2}{3}\epsilon\cos\chi)\epsilon^2 + \mathcal{O}(\epsilon^4)} \quad \text{(C.2)}$$

If we then look at the co-ordinate relations for t and r in 4.2, then we can see that:

$$dt = \frac{d\psi}{\epsilon\sqrt{\Lambda}}, \quad dr = -\frac{1}{\sqrt{\Lambda}}\epsilon\sin\chi d\chi$$

This means we can sub these expressions along with the expanded $f(r)$ in C.2 to the original metric 4.1:

$$ds^2 = -\left(1+\frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi\frac{\epsilon^2 d\psi^2}{\epsilon^2\Lambda} + \frac{1}{\Lambda}\frac{\epsilon^2\sin^2\chi d\chi^2}{(1+\frac{2}{3}\epsilon\cos\chi)\sin^2\chi\epsilon^2} + \frac{1}{\Lambda}(1+\frac{2}{3}\epsilon\cos\chi)^2 d\Omega_2$$

So to leading order in ϵ the metric is indeed given by 4.3.

C.1.1 Embedding co-ordinates

in this section we show the embedding co-ordinates 4.7 match matches the first two terms of the metric 4.6. Taking derivatives of 4.7 we see:

$$\begin{aligned}
dX_0 &= \frac{1}{\sqrt{\Lambda}}\left[\frac{-zdz}{\sqrt{1-z^2}}\sinh\psi + \sqrt{1-z^2}\cosh\psi d\right] \\
dX_1 &= \frac{1}{\sqrt{\Lambda}}\left[\frac{-zdz}{\sqrt{1-z^2}}\cosh\psi + \sqrt{1-z^2}\sinh\psi d\psi\right] \\
dX_2 &= \frac{z}{\sqrt{\Lambda}}
\end{aligned}$$

So writing the combination $-dX_0^2 + dX_1^2 + dX_2^2$:

$$\begin{aligned}
-dX_0^2 + dX_1^2 + dX_2^2 =& \frac{1}{\Lambda}\left[-\frac{z^2dz^2}{1-z^2}\sinh^2\psi + \frac{zdzd\psi}{\sqrt{1-z^2}}\cosh\psi\sinh\psi - (1-z^2)\cosh^2\psi d\psi^2 + dz^2\right. \\
&\left. + \frac{z^2dz^2}{1-z^2}\cosh^2\psi - \frac{zdzd\psi}{\sqrt{1-z^2}}\cosh\psi\sinh\psi + (1-z^2)\sinh^2\psi d\psi^2 + dz^2\right] \\
=& \frac{1}{\Lambda}\left[-(1-z^2)d\psi^2 + \frac{z^2dz^2}{1-z^2} + dz^2\right] \\
=& \frac{1}{\Lambda}\left[-(1-z^2)d\psi^2 + \frac{dz^2}{1-z^2}\right]
\end{aligned}$$

Which is exactly the first two terms in 4.6.

C.2 Action with Quantum Effects

C.2.1 Classical Trace of Stress Tensor

Here we will show that the classical consideration of a conformally invariant action leads to a traceless stress energy tensor $T_{\mu\nu}$. An action $S\left[g_{\mu\nu}, \phi, \partial_\mu\phi\right]$ (suppressing derivative for brevity) is considered conformally invariant if $S[g_{\mu\nu}, \phi, \partial_\mu\phi] = S[\tilde{g}_{\mu\nu}, \tilde{\phi}, \partial_\mu\tilde{\phi}]$ + surface integral. Where:

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \Omega(x)g_{\mu\nu}(x) \\
\tilde{\phi} &= \Omega^p(x)\phi(x)
\end{aligned}$$

Here Ω is some arbitrary scaling function and p a dimensionless constant. To see how ϕ and $g_{\mu\nu}$ vary with these transformations we can write the scaling infinitesimally as:

$$\Omega(x) \approx 1 + \lambda(x), \quad |\lambda(x)| << 1$$

With this the variations of ϕ and $g_{\mu\nu}$ are:

$$\delta\phi = \tilde{\phi} - \phi = p\lambda(x)\phi(x)$$
$$\delta g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = \lambda(x) g_{\mu\nu}$$
$$\delta(\partial_\mu\phi) = \partial_\mu\tilde{\phi} - \partial_\mu\phi = \partial_\mu[(1+p\lambda)\,\phi] - \partial_\mu\phi = \partial_\mu(p\lambda\phi) = \partial_\mu(\delta\phi)$$

With this the variation of the action $S\,[g_{\mu\nu},\phi]$ can be considered:

$$\begin{aligned} 0 = \delta S &= \int d^4x \left[\frac{\delta S}{\delta\phi}\delta\phi + \frac{\delta S}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right] \\ &= \int d^4x \left[\left(\frac{\delta S}{\delta\phi} - \partial_\mu\left[\frac{\delta S}{\delta(\partial_\mu\phi)}\right]\right)\delta\phi + \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right] \end{aligned}$$

Where we have integrated by parts. If we then assume ϕ satisfies the Euler-Lagrange equation of motion, then we are left with:

$$0 = \int d^4x \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu}\lambda(x)$$

Since $\lambda(x)$ is arbitrary, this implies that $\delta S/\delta g_{\mu\nu} = 0$, but we can recall (see [6]) that the definition of the stress energy tensor in GR is:

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{C.3}$$

This means the action being conformally invariant implies the trace of the stress energy tensor vanishes classically:

$$g_{\mu\nu}T^{\mu\nu} = 0$$

C.2.2 Path Integral Classical Limit

The path integral 4.16 leads to a natural interpretation of the classical limit of quantum mechanics. The exponent in the integral should be unitless so in SI units when $\hbar \neq 1$, the exponent is $i\frac{S}{\hbar}$ [1]. Then in the classical limit, $\hbar \to \infty$ and this integral oscillates very fast. This allows us to use the stationary phase approximation to evaluate this integral. This is the idea that fast oscillating terms such as $e^{i\frac{S}{\hbar}}$ when integrated over add up points that are located around the unit circle in the complex plane. This means the integral which adds up these points will tend towards 0 the faster this term oscillates as adding up all the points on the unit circle results in the value at its center, namely 0. The only exception to this is when the exponential happens to not be oscillating quickly. This occurs at the points where the action S does not change much when we vary the parameter the path integral is integrating over. In our case this is the field φ. These points are the only non vanishing points and hence the only points that contribute to the integral. This is nothing more then the action principle. In the classical limit the path integral implies that the only non-vanishing field configuration is the the one that extremizes the action. The same holds for any expectation values like 4.17, in the classical limit the only expectation value that survives is the energy momentum tensor for the field configuration that minimizes the action.

[1] Recall that the action has units of angular momentum, same as $\hbar$

C.2.3 Ostrogradsky Lagrangian

The purpose of this section is to derive the equations of motion for a Lagrangian density that depends on second derivatives.

One Dimensional Model

Let us first consider a one dimensional classical example for a system with a co-ordinate q. We do this to get an idea for what introducing higher order derivatives to the Lagrangian, before examining the field theory case. If we have a Lagrangian L that is a function of both q and $\dot{q}$ as well as $\ddot{q}$ ($L = L(q, \dot{q}, \ddot{q})$), then variation of the Lagrangian δL will contain a term of $\delta\ddot{q}$ [2]. i.e.:

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}$$

This means when we vary the action which is defined as $S = \int L dt$, then we must have that:

$$\delta S = \int \delta L dt = \int dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}\right]$$

We can then do the usual trick of integrating by parts and ignoring the terms which must vanish at the boundaries. We integrate the second term in the integral by parts once picking up a minus sign, but for the last term we must integrate by parts twice hence it does not change sign. This leaves us with:

$$\delta S = \int dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]\delta q$$

So enforcing the action principle that $\delta S = 0$ is equivalent to the following equations of motion:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0$$

Lagrangian Density

We now wish to extend this to the Lagrangian density. The natural extension of a Lagrangian density $\mathcal{L}(\phi, \partial_\mu)$, is $\mathcal{L}(\phi, \partial_\mu, \partial_\mu\partial_\nu\phi)$, but since we know a priori that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu, \partial^2\phi)$, where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$, we can instead proceed with this special case, as it is simpler. Following the procedure of the above one dimensional case, the variation of $\mathcal{L}(\phi, \partial_\mu, \partial^2\phi)$ is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial^2\phi)}\delta(\partial^2\phi)$$

Then just as before when we place this in the action we have to integrate by parts, with the middle term picking up a minus sign and the last staying the same. Then imposing $\delta S = 0$ is equivalent to the equation of motion:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \partial^2\left(\frac{\partial\mathcal{L}}{\partial(\partial^2\phi)}\right) = 0 \qquad \text{(C.4)}$$

[2] This is the same as expanding the total derivative of a multivariable function.

C.3 Metric Perturbation

We start by calculating the form of the $\partial^2\phi$ and $(\partial^2\phi)$ terms, up to first order in epsilon. This can be done by considering derivatives of 4.36:

$$\begin{aligned} \partial_t(e^{2\phi}) &= 2\Lambda_2\epsilon\dot{\sigma}\cos x = 2\dot{\phi}e^{2\phi} \implies \dot{\phi} = \frac{\epsilon\dot{\sigma}\cos x}{1+2\epsilon\sigma\cos x} \\ \partial_x(e^{2\phi}) &= -2\Lambda_2\epsilon\sigma\sin x = 2\phi' e^{2\phi} \implies \phi' = -\frac{\epsilon\sigma\sin x}{1+2\epsilon\sigma\cos x} \end{aligned} \tag{C.5}$$

This means that:

$$\begin{gathered} \ddot{\phi} = \frac{\epsilon\ddot{\sigma}\cos x}{1+2\epsilon\sigma\cos x} + \mathcal{O}(\epsilon^2), \quad \phi'' = -\frac{\epsilon\sigma\cos x}{1+2\epsilon\sigma\cos x} + \mathcal{O}(\epsilon^2) \\ \dot{\phi}^2 = \mathcal{O}(\epsilon^2), \quad \phi'' = \mathcal{O}(\epsilon^2) \end{gathered} \tag{C.6}$$

So we can write $\partial^2\phi$ and $(\partial^2\phi)$ as:

$$\begin{aligned} (\partial\phi)^2 &= -\dot{\phi}^2 + \phi'^2 = \mathcal{O}(\epsilon^2) \\ \partial^2\phi &= -\ddot{\phi} + \phi'' = \frac{\epsilon\ddot{\sigma}\cos x}{1+2\epsilon\sigma\cos x} - \frac{\epsilon\sigma\cos x}{1+2\epsilon\sigma\cos x} + \mathcal{O}(\epsilon^2) \\ &= -\frac{\epsilon\cos x\,[\ddot{\sigma}+\sigma]}{1+2\epsilon\sigma\cos x} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{C.7}$$

We can now use these values to plug into the equations of motion. As stated before we do not need to perturb the one sphere radius $e^{2\rho}$ as it will not enter the equations of motion to first order in ϵ. However, since the equations of motion couple ϕ and ρ, the field ρ will still be altered such that $\partial^2\rho$ does not take on the same value we had for it earlier in the non-perturbed case [3]. We will now instead use the ϕ equation of motion 4.26 to solve for $\partial^2\rho$ in terms of the other terms, containing ϕ. We can manipulate 4.26 two write:

$$\partial^2\rho = \frac{\partial^2\phi - (\partial\phi)^2 - \Lambda e^{2\rho}}{1 - \frac{w\kappa}{4}\Lambda_2\left[1+2\epsilon\sigma\cos x\right]}$$

We can then expand this to first order in ϵ using the above calculated values. $(\partial\phi)^2 = \mathcal{O}(\epsilon^2)$ so we can ignore it, all that remains is to plug in $\partial^2\phi$ from C.7 and expand the denominator using $\frac{1}{a+x} = \frac{1}{a}\left(1-\frac{x}{a}\right) + \mathcal{O}(x^2)$:

$$\begin{aligned} \partial^2\rho &= \left[-\frac{\epsilon\cos x\,[\ddot{\sigma}+\sigma]}{1+2\epsilon\sigma\cos x} - \Lambda e^{2\rho}\right] \frac{1 + \frac{\epsilon\sigma\cos x}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)}\frac{w\kappa}{2}\Lambda_2}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)} + \mathcal{O}(\epsilon^2) \\ &= \left[-\frac{\epsilon\cos x\,[\ddot{\sigma}+\sigma]}{1+2\epsilon\sigma\cos x} - \frac{\Lambda}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)}e^{2\rho} - \Lambda e^{2\rho}\frac{\epsilon\sigma\cos x}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)^2}\frac{w\kappa}{2}\Lambda_2\right] + \mathcal{O}(\epsilon^2) \end{aligned}$$

We calculate $\partial^2\rho$ as when we use the Z equation of motion 4.24 to eliminate Z from the ρ equation of motion 4.27 we are left with a term of the form $\frac{\kappa}{2}e^{2\phi}\partial^2\rho$, With the above calculation of $\partial^2\rho$ this becomes:

$$\frac{\kappa}{2}e^{2\phi}\partial^2\rho = \frac{\kappa\Lambda_2}{2}\left[-\frac{\epsilon\cos x\,[\ddot{\sigma}+\sigma]}{1+2\epsilon\sigma\cos x} - \frac{\Lambda}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)}e^{2\rho}[1+2\epsilon\sigma\cos x] - \Lambda e^{2\rho}\frac{\epsilon\sigma\cos x}{\left(1-\frac{w\kappa}{4}\Lambda_2\right)^2}\frac{w\kappa}{2}\Lambda_2\right] + \mathcal{O}(\epsilon^2)$$

[3] Recall we had $\partial^2\rho = -\Lambda_1 e^{2\rho}$

With this as well as C.7, $(\partial\phi)^2 = \mathcal{O}(\epsilon^2)$ and the ansatz $e^{2\rho} = \frac{1}{\Lambda_1 \cos^2 t}$ we can write the ρ equation of motion 4.27 as:

$$0 = \left(1 - \frac{w\kappa}{2}\Lambda_2\left[1 + 2\epsilon\sigma\cos x\right]\right)\frac{\epsilon\cos x\left[\ddot{\sigma} + \sigma\right]}{1 + 2\epsilon\sigma\cos x} - \frac{\kappa\Lambda}{2}\frac{\epsilon\cos x\left[\ddot{\sigma} + \sigma\right]}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)}$$
$$- \frac{\kappa}{2\Lambda_1}\frac{\Lambda_2}{\cos^2 t}\left[\frac{\Lambda\left[1 + 2\epsilon\sigma\cos x\right]}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)} + \frac{\Lambda\Lambda_2\frac{w\kappa}{2}\epsilon\sigma\cos x}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)}\right] + \frac{1}{\cos^2 t\Lambda_1}\left[\Lambda - \Lambda_2\left[1 + 2\epsilon\sigma\cos x\right]\right] + \mathcal{O}(\epsilon^2)$$

Some terms are not full expanded up to $\mathcal{O}(\epsilon^2)$, so doing so we have:

$$0 = \left[(1 - \frac{w\kappa}{4}\Lambda_2) - \frac{\kappa\Lambda_2}{2\left(1 - \frac{w\kappa}{4}\Lambda_2\right)}\right]\left[\ddot{\sigma} + \sigma\right]\epsilon\cos x + \frac{1}{\cos^2 t}\left[-\frac{\kappa\Lambda_2\Lambda}{2\Lambda_1}\frac{1}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)} + \frac{\Lambda}{\Lambda_1} - \frac{\Lambda_2}{\Lambda_1}\right.$$
$$\left. + \epsilon\sigma\cos x\left(-\frac{\kappa\Lambda_2\Lambda}{\Lambda_1}\frac{1}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)} - \frac{w\kappa^2\Lambda_2^2\Lambda}{4\Lambda_1}\frac{1}{\left(1 - \frac{w\kappa}{4}\Lambda_2\right)^2} - \frac{2\Lambda_2}{\Lambda_1}\right)\right] + \mathcal{O}(\epsilon^2)$$

We can then make use of some relations to simplify this expression. From the second relation between Λ_1 and Λ_2 4.33, we can write that $\left(1 - \frac{w\kappa}{4}\Lambda_2\right) = \frac{\Lambda}{\Lambda_1}$. This allows us to write our expression as:

$$0 = \left[\frac{\Lambda}{\Lambda_1} - \frac{\kappa\Lambda_2\Lambda_1}{2\Lambda}\right]\left[\ddot{\sigma} + \sigma\right]\epsilon\cos x + \frac{1}{\cos^2 t}\left[\underbrace{-\frac{\kappa\Lambda_2}{2} + \frac{\Lambda}{\Lambda_1} - \frac{\Lambda_2}{\Lambda_1}}_{0} + \epsilon\sigma\cos x\left(-\kappa\Lambda_2 - \frac{w\kappa^2\Lambda_2^2\Lambda_1}{4\Lambda} - \frac{2\Lambda_2}{\Lambda_1}\right)\right]$$

Where we have used the first relation between Λ_1 and Λ_2, 4.32 to write $\frac{\kappa\Lambda_2}{2} + \frac{\Lambda}{\Lambda_1} - \frac{\Lambda_2}{\Lambda_1} = 0$[4]. This leaves us with only terms proportional to $\epsilon\cos x$, so we can cancel them all, leaving us with, to leading order in ϵ:

$$\boxed{\frac{\ddot{\sigma}}{\sigma} = \frac{a}{\cos^2 t} - 1} \tag{C.8}$$

Where:

$$a = \frac{\kappa\Lambda_2 + \frac{w\kappa^2\Lambda_2^2\Lambda_1}{4\Lambda} + \frac{2\Lambda_2}{\Lambda_1}}{\frac{\Lambda}{\Lambda_1} - \frac{\kappa\Lambda_2\Lambda_1}{2\Lambda}}$$

We can feed this along with the definitions of Λ_1 and Λ_2 4.34 and 4.35 into Mathematica, which tells us that a is given by 4.38. Matching the result found in Bousso and Hawking's paper.

[4] If one also perturbs the one sphere radius $e^{2\rho}$, they would find that to first order in epsilon the perturbation is proportional to this equation, and hence does not contribute to the equation of motion.

Dimension Reduction of Ricci scalar

In[1]:=
```
InverseMetric[g_] := Simplify[Inverse[g]]
ChristoffelSymbol[g_, xx_] := Block[{n, ig, res}, n = Length[g];
  ig = InverseMetric[g];
  res = Table[(1/2) * Sum[ig[[i, s]] * (-D[g[[j, k]], xx[[s]]] + D[g[[j, s]], xx[[k]]] + D[g[[s, k]], xx[[j]]]), {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}];
  Simplify[res]]
RiemannTensor[g_, xx_] := Block[{n, Chr, res}, n = Length[g];
  Chr = ChristoffelSymbol[g, xx];
  res = Table[D[Chr[[i, k, m]], xx[[l]]] - D[Chr[[i, k, l]], xx[[m]]] + Sum[Chr[[i, s, l]] * Chr[[s, k, m]], {s, 1, n}] -
      Sum[Chr[[i, s, m]] * Chr[[s, k, l]], {s, 1, n}], {i, 1, n}, {k, 1, n}, {l, 1, n}, {m, 1, n}];
  Simplify[res]]
RicciTensor[g_, xx_] := Block[{Rie, res, n}, n = Length[g];
  Rie = RiemannTensor[g, xx];
  res = Table[Sum[Rie[[s, i, s, j]], {s, 1, n}], {i, 1, n}, {j, 1, n}];
  Simplify[res]]
RicciScalar[g_, xx_] := Block[{Ricc, ig, res, n}, n = Length[g];
  Ricc = RicciTensor[g, xx];
  ig = InverseMetric[g];
  res = Sum[ig[[s, i]] × Ricc[[s, i]], {s, 1, n}, {i, 1, n}];
  Simplify[res]]
```

4D-Metric:

In[6]:=
```
xx = {t, x, θ, ψ};
g = {{-E^(2 ρ[x, t]), 0, 0, 0}, {0, E^(2 ρ[x, t]), 0, 0},
    {0, 0, E^(-2 ϕ[x, t]), 0}, {0, 0, 0, E^(-2 ϕ [x, t]) Sin[θ]^2}};
RicciScalar[g, xx] // Expand
```

Out[8]=

$$2\, e^{-2\rho[x,t]+2(\rho[x,t]+\phi[x,t])} + 6\, e^{-2\rho[x,t]}\, \phi^{(0,1)}[x,\, t]^2 + 2\, e^{-2\rho[x,t]}\, \rho^{(0,2)}[x,\, t] - 4\, e^{-2\rho[x,t]}\, \phi^{(0,2)}[x,\, t] - 6\, e^{-2\rho[x,t]}\, \phi^{(1,0)}[x,\, t]^2 - 2\, e^{-2\rho[x,t]}\, \rho^{(2,0)}[x,\, t] + 4\, e^{-2\rho[x,t]}\, \phi^{(2,0)}[x,\, t]$$

2D Metric

In[9]:=
```
xx = {t, x};  g = {{-E^(2 ρ[x, t]), 0}, {0, E^(2 ρ[x, t])}};
RicciScalar[g, xx] // Expand
```

Out[10]=

$$2\, e^{-2\rho[x,t]}\, \rho^{(0,2)}[x,\, t] - 2\, e^{-2\rho[x,t]}\, \rho^{(2,0)}[x,\, t]$$

Bibliography

[1] Jacob D Bekenstein. “Black holes and entropy”. In: *Physical Review D* 7.8 (1973), p. 2333.

[2] Raphael Bousso and Stephen Hawking. “Trace anomaly of dilaton-coupled scalars in two dimensions”. In: *Physical Review D* 56.12 (1997), p. 7788.

[3] Raphael Bousso and Stephen W Hawking. “(Anti-) evaporation of Schwarzschild–de Sitter black holes”. In: *Physical Review D* 57.4 (1998), p. 2436. DOI: 10.1103/PhysRevD.57.2436.

[4] Thomas Brosnan. *Quantum Field Theory I*. Lecture notes, Trinity College Dublin. 2024. URL: https://tbrosnan12.github.io/documents/Fourth_year/First_semester/Quantum_Field_Theory_I.pdf.

[5] C. G. Callan et al. “Evanescent black holes”. In: *Phys. Rev. D* 45 (1992), p. 1005. DOI: 10.1103/PhysRevD.45.1005. eprint: hep-th/9111056.

[6] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019. DOI: 10.1017/9781108770385.

[7] Steven M Christensen and Stephen A Fulling. “Trace anomalies and the Hawking effect”. In: *Physical Review D* 15.8 (1977), p. 2088.

[8] Brian P Dolan. “The definition of mass in asymptotically de Sitter space-times”. In: *Classical and Quantum Gravity* 36.7 (2019), p. 077001.

[9] Ernesto Frodden and Nicolás Valdés. “Unruh Effect: Introductory notes to quantum effects for accelerated observers”. In: *International Journal of Modern Physics A* 33.27 (Sept. 2018), p. 1830026. ISSN: 1793-656X. DOI: 10.1142/s0217751x18300260.

[10] Steven B Giddings. “Quantum mechanics of black holes”. In: *arXiv preprint hep-th/9412138* (1994).

[11] Paul Ginsparg and Malcolm J Perry. “Semiclassical perdurance of de Sitter space”. In: *Nuclear Physics B* 222.2 (1983), pp. 245–268.

[12] Thomas Hartman. “Lecture notes on classical de Sitter space”. In: *arXiv preprint arXiv:1205.3855* (2017).

[13] Stephen W Hawking. “Particle creation by black holes”. In: *Communications in mathematical physics* 43.3 (1975), pp. 199–220.

[14] Justin D Hayward. “Entropy in the Russo-Susskind-Thorlacius model”. In: *Physical Review D* 52.4 (1995), p. 2239.

[15] Leonard Parker and David Toms. *Quantum field theory in curved spacetime: quantized fields and gravity*. Cambridge university press, 2009. DOI: 10.1017/CBO9780511813924.

[16] Walter Rudin. *Real and complex analysis*. McGraw-Hill, Inc., 1987.

[17] J. G. Russo, L. Susskind, and L. Thorlacius. “Black hole evaporation in 1 + 1 dimensions”. In: *Phys. Lett. B* 292 (1992), p. 13. DOI: 10.1016/0370-2693(92)90856-M. eprint: hep-th/9201074.

[18] William G Unruh. “Notes on black-hole evaporation”. In: *Physical Review D* 14.4 (1976), p. 870.