

# Calculus on Manifolds

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”  $p \in N$  is ”

-Florian

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# 1 Topology on $\mathbb{R}^n$

## 1.1 Metric space

- Let  $X$  be a set, A *metric* on a set is a function that measures distances  $d : X \times X \rightarrow \mathbb{R}$ . It has the following properties:

$$\begin{aligned} d(x, y) &= d(y, x) \\ d(x, y) &\geq 0 \\ d(x, y) &= 0 \text{ iff } x = y \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned} \tag{1.1}$$

$(X, d)$  together make a *metric space*.

- Any subset  $Y \subset X$  is itself a metric space with  $d(x, y) \Big|_{Y \times Y}$  (restricted to  $Y$ ).

## 1.2 Open/Closed

- Let  $(X, d)$  be a metric space  $U \subset X$  is *open* if  $\forall p \in U, \exists \epsilon > 0$  st.  $B_\epsilon(p) := \{x \in X | d(x, p) \leq \epsilon\}$  and *closed* if  $X - U$  (the compliment set) is open.
- If we have  $U \subset Y \subset X$ ,  $(X, d)$  a metric space, for us in all applications  $X = \mathbb{R}^n$ .  $U$  is open/closed in  $(Y, d|_{Y \times Y}) \iff \exists V \subset X$  open/closed st.  $U = V \cap Y$ .

## 1.3 Continuity

- If we have  $f : X \rightarrow Y$ , with  $X$  and  $Y$  metric spaces, is *continuous* if,  $f^{-1}(U)$  is open with  $U \subset Y$  is open.

If  $f : X \rightarrow Y$  is a bijection, continuous and  $f^{-1}$  continuous we call  $f$  a homomorphism.

## 1.4 Compact

- $X$  is *compact* if every open cover has a finite subcover, i.e.  $\forall \{U_\alpha\}_{\alpha \in I}, U_\alpha \subset X$  ( $U_\alpha$  open) st.  $X \subset \bigcup_{\alpha \in I} U_\alpha$ , then  $\exists \alpha_1, \dots, \alpha_k \in I$  st.  $X \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$ .

## 1.5 Heine Boral theorem

- $X \subset \mathbb{R}^n$  is compact if bounded ( $\exists R \in \mathbb{R}$  st  $X \subset B_R(0)$ ) and closed in  $\mathbb{R}^n$ .

## 1.6 Differentiation

- $f : U \rightarrow V$ , ( $U \subset \mathbb{R}^n$ ,  $V \subset \mathbb{R}^m$ ) is differentiable at  $p \in U$  with *derivative*  $Df(p) \in Mat(m, n)$  if:

$$\lim_{x \rightarrow p} \frac{f(x) - f(p) - Df(p)(x - p)}{\|x - p\|} = 0 \tag{1.2}$$

- $f$  is (of class)  $C^1$  if it is differentiable at all  $p \in U$  and  $Df : U \rightarrow \text{Mat}(m, n) \cong \mathbb{R}^{mn}$  is continuous.
- $f$  is  $C^r$  if  $Df$  is  $C^{r-1}$ ,  $f$  is *smooth* or  $C^\infty$  if it is  $C^t \forall t > 0$ .
- If we have  $f : U \rightarrow \mathbb{R}^m$ , ( $U \subset \mathbb{R}^n$ ). Then  $x \mapsto (f_1(x), \dots, f_m(x))$  is  $C^r$ , if:

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_j : U \rightarrow \mathbb{R} \quad (1.3)$$

Exists, and is continuous for all  $k \in \{1, \dots, r\}$ ,  $i_1, \dots, i_k \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ . In which case the derivative can then be expressed as:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (1.4)$$

### 1.7 Chain rule

- Consider  $U \xrightarrow{g} V \xrightarrow{f} W$ , where  $f$  and  $g$  are differentiable (or  $C^r$ ), then so is  $f \circ g$  and:

$$D(f \circ g)(x) = Df(g(x)) \cdot Dg(x) \quad (1.5)$$

This is the chain rule and the  $\cdot$  here refers to matrix multiplication.

### 1.8 Diffeomorphism

- If we have  $f : U \rightarrow V$ , a smooth bijection and  $U, V$  open (in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively) st.  $f^{-1}V \rightarrow U$  exists and is also smooth. Then we call  $f$  a diffeomorphism.

### 1.9 Inverse function Theorem

- Let  $f : V \rightarrow \mathbb{R}^n$  be  $C^r$  ( $1 \leq r \leq \infty$ ) and  $V \subset \mathbb{R}^n$ . For  $p \in V$ , suppose  $Df(p)$  is non-singular (i.e. an invertible  $n \times n$  matrix  $\iff \det(Df) \neq 0$ ). Then  $\exists p \in U \subset V$ ,  $U$  open, st,
  - $f|_U : U \rightarrow f(U)$ , is a  $C^r$ -diffeomorphism. i.e.  $f|_U : U \rightarrow f(U)$  is a bijection
  - $f(U)$  is open
  - $f^{-1}|_U : U \rightarrow f(U)$  is  $C^r$ .

## 2 Manifolds

- "Slogan" (informal definition)  $M \subset \mathbb{R}^n$  is a manifold if it is "smooth" without corners/intersections

### 2.1 Manifolds

- Let  $d > 0$ ,  $M \subset \mathbb{R}^n$  is a smooth/ $C^r$  manifold of dimension  $d$  if  $\forall p \in M$ ,  $\exists p \in V \subset M$ ,  $U \subset \mathbb{R}^d$  ( $V$  and  $U$  open) and  $\alpha : U \rightarrow V$ , st:
  - $\alpha$  is smooth/ $C^r$
  - $\alpha$  is a bijection with a continuous inverse (  $\iff$  is a homeomorphism)
  - $D\alpha(x)$  has Rank  $d$ .

We will see this means  $\alpha$  is a diffeomorphism.

#### 2.1.1 Parameterised manifold

- Sometimes a only a single function  $\alpha : U \rightarrow M$  is needed in the definition of a manifold. In this case we call  $(M, \alpha)$  a parameterized manifold.

From now on we will only discuss smooth/ $C^\infty$  manifolds

### 2.2 Alternate definitions

- If we have a set  $M \subset \mathbb{R}^n$ ,  $d > 0$ ,  $p \in M$ . Then the following are equivalent:
  - $\exists p \in V \subset M$ ,  $U \subset \mathbb{R}^d$ , ( $V$  and  $U$  open),  $\alpha : U \rightarrow V$  a smooth homomorphism, st,  $D\alpha(x)$  has rank  $d$ ,  $\forall x \in U$ .
  - $\exists p \in V \subset \mathbb{R}^n$ ,  $U \subset \mathbb{R}^n$  ( $V$  and  $U$  open),  $\beta : U \rightarrow V$  a diffeomorphism and  $\beta(U \cap (\mathbb{R}^d \times \{0\})) = V \cap M$ .
- This second definition is new and the set  $U \cap (\mathbb{R}^d \times \{0\})$  is just the intersection of  $U \subset \mathbb{R}^n$  and the space  $\mathbb{R}^d$  extended into  $\mathbb{R}^n$  by adding 0 to the  $d$  dimensional tuples  $n - d$  times until they become  $\mathbb{R}^n$ . This is effectively saying we want to be able to straighten out manifold neighbourhoods.

### 2.3 Locally smooth

- We want to say a  $d$ - manifold looks locally like  $\mathbb{R}^d$ .
- Let  $M \subset \mathbb{R}^n$ ,  $N \subset \mathbb{R}^m$  be subsets. A function  $f : M \rightarrow N$  is smooth if  $\exists M \subset V \subset \mathbb{R}^n$  ( $V$  open) and  $\tilde{f} : V \rightarrow \mathbb{R}^m$  smooth st,  $\tilde{f}|_M = f$  and  $f : M \rightarrow N$  is a diffeomorphism (It is a smooth bijection and has a smooth inverse). Note that we do not require  $\tilde{f}$  to have  $\tilde{f}^{-1} \circ \tilde{f} = \mathbb{I}$ .

It follows that we can say:  $f : M \rightarrow N$  a diffeomorphism,  $A \subset M \implies f|_A : A \rightarrow f(A)$  is a diffeomorphism.

- Remark These two facts are used in the proof of the following theorem. This theorem looks exactly like the definition of a manifold but note the swapping of  $V$  and  $U$ , which changes the statement to that the condition for a manifold is that there is a smooth mapping from the manifold to  $\mathbb{R}^d$ .



**2.3.1 Local smoothness definition of a manifold**

- Let  $d > 0$ ,  $M \subset \mathbb{R}^n$  is a smooth/ $C^r$  manifold of dimension  $d$  if  $\forall p \in M$ ,  $\exists p \in V \subset M$ ,  $U \subset \mathbb{R}^d$  ( $V$  and  $U$  open) and  $\alpha : V \rightarrow U$ , st  $\alpha$  is a diffeomorphism.

## 3 Partitions of unity

### 3.1 Main idea

- Given  $\{U_i\}_{i \in I}$  a partition of unity is a collection of smooth functions  $\{\psi_i\}$ ,  $\psi_i : \mathbb{R}^n \rightarrow [0, \infty)$  a diffeomorphism st  $\{x | \psi_i(x) \neq 0\} \subset U_i$  st  $\sum_{i \in I} \psi_i(x) = 1$ .
- We have a local definition of a manifold and we want to extend it so that we have one single function smooth across all of  $M$ .

### 3.2 Theorem

- Let  $\mathbb{R} \supset V = \bigcup_{\alpha \in A} U_\alpha$ , where  $U_\alpha$  are open, then there exists  $\phi_1, \phi_2, \dots : V \rightarrow [0, 1]$ , st:
    - For each  $i \in \mathbb{N} \exists \alpha \in A$  st  $S_i := \text{supp}(\phi_i) = \overline{\{x \in V | \psi_i(x) \neq 0\}} \subset U_\alpha$
    - Each  $p \in A$  has a neighbourhood intersecting finitely many  $S_i$ 's.
    - $\sum_{i \in I} \psi_i(x) = 1, \forall x \in V$ .
    - $S_i$ 's are compact.
    - $\psi_i$  are smooth.
- $\{\psi_i\}$  is called a partition of unity subordinate to  $\{U_\alpha\}$ .

### 3.3 Lemma 1

- $\{U_\alpha\}$  as above, then  $\exists p_1, p_2, \dots \in \mathbb{R}^n, \epsilon_1, \epsilon_2, \dots \in \mathbb{R}_{>0}$  st:
  - $\bigcup_{i=1} B_{\epsilon_i}(p_i) = V$
  - Each  $B_{2\epsilon_i}(p_i)$  is contained in a  $U_\alpha$ .
  - Each point  $p \in V$  has a neighbourhood intersecting finitely many  $B_{2\epsilon_i}(p_i)$ .

### 3.4 Sub-lemma

- One can find  $k_1 \subset k_2 \subset \dots \subset V$  st:
  - $k_i$  are compact.
  - $k_i \subset \overset{\circ}{k}_{i+1}$
  - $\bigcup_{i=1} k_i = V$

### 3.5 Lemma 2

- Let  $p \in \mathbb{R}^n, \epsilon > 0$  Then  $\exists \psi : \mathbb{R}^n \rightarrow [0, 1]$  st:
  - $\psi$  smooth.
  - $\text{supp}(\psi) \subset B_{2\epsilon}(p)$
  - $\psi > 0$ , on  $B_\epsilon(p)$

**3.6 Extension of locally smooth functions**

- Let  $M \subset \mathbb{R}^n$  a subset,  $f : M \rightarrow \mathbb{R}^m$ . Suppose  $f$  is locally smooth, i.e.  $\forall p \in M \exists V \subset M$ , st  $f|_V : V \rightarrow \mathbb{R}^m$  is smooth, Then  $f$  is smooth on  $M$ .

This theorem is proved using partitions of unity.

## 4 Boundary of manifolds

### 4.1 Upper half plane

- We define the upper half plane in  $\mathbb{R}^d$  to be:  $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0}$ , that is:

$$\mathbb{H} = \{(x_1, x_2, \dots, x_d) | x_d \geq 0\} \quad (4.1)$$

- The boundary of this plane is then defined as:  $\partial\mathbb{H} := \mathbb{R}^{d-1} \times 0 \subset \mathbb{H}$ . We then have that  $\mathring{\mathbb{H}} := \mathbb{H} \setminus \partial\mathbb{H}$ .

### 4.2 Boundary of a manifold

- A subset  $M \subset \mathbb{R}^n$  is a  $d$ -manifold with a boundary if it is basically diffeomorphic to open subsets of  $\mathbb{H}^d$ . that is  $\forall p \in M \exists p \in V \subset M$ ,  $U \subset \mathbb{H}^d$  ( $V$  and  $U$  open) and  $\alpha : U \rightarrow V$  a diffeomorphism.

#### 4.2.1 Proposition

- The condition is equivalent to  $\alpha$  being a smooth homomorphism and  $D\alpha(x)$  being of rank  $d \forall x \in U$ .

### 4.3 Lemma

- If we have  $\mathbb{H}^d \supset U \xrightarrow{\alpha} \mathbb{R}^n$  smooth with extensions  $\tilde{\alpha} : \tilde{U} \rightarrow \mathbb{R}^n$ , (here  $U \subset \tilde{U}$ ). Then  $D\tilde{\alpha}(x) \forall x \in U$  does not depend on the extension.
  - For  $x \in \mathring{\mathbb{H}}^d$ ,  $D\tilde{\alpha}(x) = D\alpha(x)$
  - For  $x \in \partial\mathbb{H}^d \cap U$ ,  $D\tilde{\alpha}(x) = \left( \frac{\partial \tilde{\alpha}_i(x)}{\partial x_j} \right)_{i,j}$ . Where for  $j \neq d$  this derivative is defined in the normal way, but for  $j = d$ , instead of having a two sided limit in the definition we use a one sided limit, from the side that is in the half-plane.

$$\frac{\partial \tilde{\alpha}_i(x)}{\partial x_d} = \lim_{\epsilon \rightarrow 0^+} \frac{\tilde{\alpha}_i(x + \epsilon e_d) - \tilde{\alpha}_i(x)}{\epsilon} \quad (4.2)$$

### 4.4 Change of co-ordinates transformation

- Let  $M$  be a manifold with a boundary and  $\alpha_i : V_i \rightarrow U_i$ ,  $i = 1, 2$ , two co-ordinate patches. Then  $\alpha_2^{-1} \circ \alpha_1 : \alpha_1^{-1}(U_1 \cap U_2) \rightarrow \alpha_2^{-1}(U_1 \cap U_2)$  is a diffeomorphism.

This is essentially saying we should be able to map smoothly between the pre-images of the co-ordinate patches that map to the same part of the manifold.

### 4.5 Interior and Boundary points

- Let  $M$  be a manifold with a boundary.

- We call  $p \in M$  an interior point if  $\exists \alpha : U \rightarrow V$  a co-ordinate patch, st,  $p = \alpha(x)$ ,  $\forall x \in \mathring{\mathbb{H}}^d \cap U$ . Then we can define:

$$\mathring{M} = \{x \in M | x \text{ is an interior point} \} \quad (4.3)$$

- We call  $p \in M$  an boundary point if  $\exists \alpha : U \rightarrow V$  a co-ordinate patch, st,  $p = \alpha(x)$ ,  $\forall x \in \partial \mathbb{H}^d \cap U$ .

$$\partial M = \{x \in M | x \text{ is an boundary point} \} \quad (4.4)$$

- Warning: These definitions are not the same as in topology.  $\mathring{M}$  is not equal to the topological interior of  $M$  in  $\mathbb{R}^n$  and the same for  $\partial M$ .

#### 4.6 Boundary of manifold is manifold

- let  $M$  be a  $d$ -manifold with a boundary. Then  $\partial M$  is a  $(d - 1)$ -manifold with a boundary.

#### 4.7 Lemma

- $M = \mathring{M} \sqcup \partial M$ .

(Disjoint union, a union with the additional information that the sets don't have any elements in common).

#### 4.8 Manifolds from functions

- Let  $f : \mathbb{R}^n \supset U \rightarrow \mathbb{R}$  be a smooth function ( $U$  open), we define:

$$M = \{x \in U | f(x) = 0\} = f^{-1}(\{0\}). \quad (4.5)$$

And:

$$N = \{x \in U | f(x) \geq 0\} = f^{-1}([0, \infty)). \quad (4.6)$$

Suppose that  $\forall x \in M$ ,  $Df(x)$  has rank 1, i.e.  $Df(x) \neq 0$ , then  $N$  is a manifold with boundary  $\partial N = M$ .

## 5 Tangent spaces

### 5.1 Tangent spaces

- Let  $M \in \mathbb{R}^n$  be a manifold with a boundary,  $p \in M$ ,  $\alpha : U \rightarrow V$  be a chart around  $p$ ,  $x_0 \in U$  be st  $\alpha(x_0) = p$ . The *tangent space* of  $M$  at  $p$  is:

$$T_p M := \text{Image}(D\alpha(x_0)) \subset \mathbb{R}^n \quad (5.1)$$

#### 5.1.1 Lemma

- This definition does not depend on  $\alpha$ .

### 5.2 Maps Between tangent spaces

- Let  $M, N$  be manifolds with boundaries and  $f : M \rightarrow N$  a smooth map. Then  $Df(p) = D\tilde{f}(p)$ , for some extension  $\tilde{f}$  of  $f$ , defines a linear map  $D\tilde{f}(p) : T_p M \rightarrow T_{f(p)} N$  for all  $p \in M$ .

### 5.3 Tangent Bundle

- Let  $m \subset \mathbb{R}^n$  be a manifold with a boundary. then the *Tangent Bundle* of  $M$  is defined as the disjoint union of all the tangent spaces:  $TM = \bigsqcup_{p \in M} T_p M$ . i.e.:

$$TM = \{(x, v) \in M \times \mathbb{R}^n | v \in T_x M\} \quad (5.2)$$

We then have that:

- $TM$  is a  $2d$ -manifold with a boundary.
- $f : M \rightarrow N$  smooth  $\implies \tilde{f}(p) : TM \rightarrow TN$  i.e.  $(p, v) \rightarrow (f(p), Df(p)v)$  is smooth.
- if we have  $M \xrightarrow{f} N \xrightarrow{g} L$  smooth  $\implies D(g \circ f) = Dg \circ Df$ , (chain rule).

### 5.4 Regular and Critical values

- let  $f : M \rightarrow N$  be smooth, we say  $p \in N$  is a *regular value* if  $Df(x) : T_x M \rightarrow T_p N$  is *onto* (surjective)  $\forall x \in f^{-1}(\{p\})$ , otherwise we call  $p$  a *critical point*.

#### 5.4.1 Regular value manifold

- If we have  $f : M \rightarrow N$  be smooth,  $\partial M = \emptyset = \partial N$  and  $p \in N$  a regular value. Then  $L = f^{-1}(\{p\})$  is a manifold. Moreover,  $T_x L = \ker(Df(x) : T_x M \rightarrow T_p N)$ .
  - Remark: We can find cases where this doesn't work. For example for  $f(x, y, z) = z - xy \geq 0$ , 0 is a regular point ( $Df = (-y, -x, 1)$ ) but the corresponding  $L = f^{-1}(\{p\})$  is not a manifold with a boundary as  $\partial L$  does not have 0 as a regular point for  $\partial L = (z - xy, z) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ . To fix this we just have to restrict the boundary of  $L$  to  $\partial L = f^{-1}(\{0\}) \cap \partial M$ .

**5.5 Sard's Theorem**

- Let  $f : M \rightarrow N$  be smooth. Then the set of critical values  $\text{crit}(f) \subset N$  has "*measure zero*". In particular  $\{p \in N \mid p \text{ regular value of } f\}$  is dense in  $N$ .

## 6 Multi-linear Algebra

- let  $V$  be a vector space. A function  $T : V^k \rightarrow \mathbb{R}$  is called multi-linear, (or a  $k$  tensor), if for  $v_1, \dots, v_i, \dots, v_k \in T$ , the function  $v \rightarrow T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$  is linear. This just means it is linear in each variable.
- The space of all such functions is denoted  $\mathcal{L}^k(V)$ , so:

$$\mathcal{L}^k(V) := \{T : V^k \rightarrow \mathbb{R} \mid T \text{ multilinear}\} \quad (6.1)$$

Usually we denote  $\mathcal{L}^1(V) = V^*$  and  $\mathcal{L}^0(V) = \{0\}$ . We can then show that  $\mathcal{L}^k(V)$  is a vector space with  $(\lambda f + g)(v_1, \dots, v_k) = \lambda f(v_1, \dots, v_k) + g(v_1, \dots, v_k)$ ,  $\lambda \in \mathbb{R}$ .

### 6.1 Basis vectors

- Let  $e_i$  be a basis of  $V$ , we define  $e^j \in V^*$ , via  $e^j v_i = e^j \sum_i a_i e_i = a_j$ . These form what More generally for  $I = (i_1, \dots, i_k)$ , we defined  $e^I(v_1, \dots, v_k) = e^{i_1}(v_1) \cdot \dots \cdot e^{i_k}(v_k)$ .
- The set  $\{e^I\}, I \in \{1, \dots, d\}^k$  forms a basis of  $\mathcal{L}^k(V)$ . In the particular  $\dim \mathcal{L}^k(V) = (\dim V)^k$ .

### 6.2 Tensor product

- let  $f \in \mathcal{L}^k(V)$  and  $g \in \mathcal{L}^l(V)$ , we define the following operation  $f \otimes g \in \mathcal{L}^{k+l}(V)$ , by:

$$f \otimes g(v_1, \dots, v_{k+l}) = f(v_1, \dots, v_k) \cdot g(v_{k+1}, \dots, v_{k+l}) \quad (6.2)$$

This is the *tensor product* and has the following properties. let  $f, g$  and  $h$  be tensors, then:

$$\begin{aligned} f \otimes (g \otimes h) &= (f \otimes g) \otimes h \\ (\lambda f) \otimes g &= \lambda(f \otimes g) = f \otimes (\lambda g) \\ (f + g) \otimes h &= f \otimes h + g \otimes h, \quad h \otimes (f + g) = h \otimes f + h \otimes g \\ e^I &= e^{i_1} \otimes \dots \otimes e^{i_k} \end{aligned} \quad (6.3)$$

### 6.3 Dual transformation

- Let  $A : V \rightarrow W$  be a linear map. We define the dual transformation,  $A^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$  by :

$$(A^* f)(v_1, \dots, v_k) = f(Av_1, \dots, Av_k) \quad (6.4)$$



- It can then be shown that the dual transformation has the following properties:

$$\begin{aligned} A^* \text{ is linear} \\ A^*(f \otimes g) &= A^*f \otimes A^*g \\ (A \cdot B)^* &= B^* \cdot A^* \end{aligned} \tag{6.5}$$

## 7 Alternating Tensors

### 7.1 Symmetric/Alternating tensors

- A tensor  $f \in \mathcal{L}^k(V)$  is called:
  - *symmetric* if  $f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = f(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$ .
  - *alternating* if  $f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -f(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$ . We let  $S^k V$  and  $\mathcal{A}^k V$  denote the vector space of symmetric/alternating respectively.

### 7.2 Symmetric group

- The permutation or symmetric group is defined as:

$$S_k = \{\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\} | \sigma \text{ a bijection}\} \quad (7.1)$$

Since it is a group it also follows that for  $\sigma, \tau \in S_k$ , then  $\sigma \circ \tau, \sigma^{-1} \in S_k$

#### 7.2.1 Elementary permutation

- An elementary permutation is defined as:

$$e_i(l) = \begin{cases} i+1, & l = i \\ i, & l = i+1 \\ l, & \text{otherwise} \end{cases} \quad (7.2)$$

#### 7.2.2 Lemma

- Every  $\sigma$  is a composite of the elementary permutations  $e_i$ .

### 7.3 Sign function

- There exists a function,  $\text{sgn} : S_n \rightarrow \{\pm 1\}$  st:
  - $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$
  - $\text{sgn}(e_i) = -1$
  - $\text{sgn}(\sigma) = (-1)^m$ , if  $\sigma$  is made of  $m$  elementary permutations.
  - $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$
  - $\text{sgn}(\sigma) = -1$ , if  $\sigma$  keeps  $p \neq q$  fixed and keeps everything else fixed.

Moreover, the first and second property here uniquely determine  $\text{sgn}$ .

## 7.4 Permutation of tensors

- If we have  $f \in \mathcal{L}^k(V)$ ,  $\sigma \in S_k$ , then we can define the following:

$$f^\sigma(v_1, \dots, v_k) := f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \quad (7.3)$$

### 7.4.1 Lemma

- $\mathcal{L}^k(V)$  is a linear  $S_k$ -representation if for  $f \in \mathcal{L}^k(V)$ ,  $f^{\sigma\tau} = (f^\tau)^\sigma$ ,  $f^\sigma = f$  (i.e. it is symmetric) and  $f \mapsto f^\sigma$  is a linear map from  $\mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ .

## 7.5 Sgn definition of tensors

- For  $f \in \mathcal{L}^k(V)$ ,  $\sigma \in S_k$ 
  - $f$  is symmetric iff  $f^\sigma = f$ ,  $\forall \sigma \in S_k$
  - $f$  is alternating iff  $f^\sigma = \text{sgn}(\sigma)f$ ,  $\forall \sigma \in S_k$

### 7.5.1 Lemma

- $f \in \mathcal{L}^k(V)$  is alternating iff  $f(v_1, \dots, v_k) = 0$ , whenever  $v_i = v_j$ , for some  $i \neq j$ .

### 7.5.2 Lemma

- Let  $f \in \mathcal{A}^k(V)$  and suppose  $f(e_{i_1}, \dots, e_{i_k}) = 0$ ,  $\forall (i_1 \leq \dots \leq i_k)$ , then  $f = 0$ .

## 7.6 Alternating Tensor

- Let  $I = (i_1 \leq \dots \leq i_k)$ , we define a unique k-tensor  $\psi_I$  as:

$$\psi_I = \sum_{\sigma} \text{sgn}(\sigma) (e^I)^\sigma \quad (7.4)$$

This acts on a set of basis vectors  $e_{j_1}, \dots, e_{j_k}$  as follows:

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & \text{if } (j_1, \dots, j_k) = (i_1, \dots, i_k) \\ 0, & \text{otherwise} \end{cases} \quad (7.5)$$

This is because  $(e^I)(e_{j_1}, \dots, e_{j_k})$  is defined to act by:  $(e^{i_1} e_{j_1})(e^{i_2} e_{j_2}) \cdots (e^{i_k} e_{j_k})$ .

## 7.7 Basis of Alternating tensors

- $\{\psi_I\}$ , with  $I$  ascending, form a basis for  $\mathcal{A}^k(V)$ . In particular  $\dim \mathcal{A}^k(V) = \binom{n}{k}$ , where  $n = \dim V$ .

- This is because we can write any alternating tensor  $f \in \mathcal{A}^k(V)$  in terms of these tensors. Consider  $g = \sum_J d_J \psi_J$ , where  $d_J = f(e_{j_1}, \dots, e_{j_k})$  (output of a tensor so just a scalar) and  $J$  is all ascending indices of order  $k$ . Then the action of this new function  $g$  on the basis vectors  $g(e_{i_1}, \dots, e_{i_k}) = d_I \cdot (1) = f(e_{i_1}, \dots, e_{i_k})$ , so we can say that  $g = f$  and thus any alternating tensor  $f$  can be expanded over  $\psi_I$ .
- It can also be noted that if  $k = \dim V \implies \dim \mathcal{A}^k(V) = 1$ .

- This allows us to write:

$$\mathcal{A}^k(V) = \{\lambda \psi^{(1,2,\dots,n)} | \lambda \in \mathbb{R}\} \quad (7.6)$$

## 7.8 Alternating Dual

- Let  $B : V \rightarrow W$  be a linear transformation, If  $f$  is an alternating tensor, then  $B^*f$  is also an alternating tensor.

## 7.9 Alternating dual

- Let  $B : V \rightarrow W$  be a linear map, then  $B^*$  restricted to  $B^* : \mathcal{A}^k(W) \rightarrow \mathcal{A}^k(V)$ , that is  $B^*f$  is alternating if  $f$  is.

### 7.9.1 Dual determinant

- For  $B : V \rightarrow V$  and  $k = \dim V = n$ , then we have that:

$$B^*f = \det(B)f, \quad f \in \mathcal{A}^k(V) \quad (7.7)$$

## 8 The wedge product

- The motivation behind this is that we would like to be able to combine alternating tensors in such a way so that the result is also an alternating tensor!

### 8.1 The Wedge product

- $\exists$  an operation  $\mathcal{A}^k(V) \times \mathcal{A}^l(V) \rightarrow \mathcal{A}^{k+l}(V)$   $((f, g) \mapsto f \wedge g)$  satisfying the following:
  - $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
  - $(f, g) \mapsto f \wedge g$  is bilinear,  $(f + \lambda g) \wedge h = f \wedge h + \lambda(g \wedge h)$
  - $f \wedge g = (-1)^{kl} g \wedge f$
  - $\psi^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$  for a basis  $e_i$  of  $V$ ,  $I = (i_1 \leq \dots \leq i_k)$ .

The wedge product is uniquely defined by these for properties, furthermore let  $T : V \rightarrow W$  be a linear map, then:

$$T^*(f \wedge g) = T^*f \wedge T^*g \in \mathcal{A}^{k+l}(V) \quad (8.1)$$

### 8.2 Alternating algebra

- The direct sum  $\bigoplus_{k=0}^{\infty} \mathcal{A}^k(V)$  form an associative, graded (anti-symmetric) commutative algebra, module in  $V$  st:  $e^{i_1} \wedge \cdots \wedge e^{i_k} = \psi^I$ .

### 8.3 Form of the wedge product

- So far we have just said there exists a wedge product but what does it actually look like? To do this we have to define a specific operator:

#### 8.3.1 Averaging operator

- This is  $A : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$  and acts by:

$$Af := \sum_{\sigma \in S_k} \text{sgn}(\sigma) f^\sigma \quad (8.2)$$

This operator satisfies that:

- $A$  is linear
- $Af \in \mathcal{A}^k(V)$
- If  $f \in \mathcal{A}^k(V)$ , then  $Af = k!f$
- This then allows us to define the wedge product for  $l \in \mathcal{A}^k(V)$  and  $g \in \mathcal{A}^l(V)$ :

$$f \wedge g := \frac{1}{k!l!} A(f \otimes g) \quad (8.3)$$

## 9 Differential forms

- Let  $M \subset \mathbb{R}^n$  be a manifold with a boundary. A differential form of order/degree  $k$  is a smooth function:

$$\omega : \{(p, v_1, \dots, v_k) \in M \times \mathbb{R}^n \times \dots \times \mathbb{R}^n | v_i \in T_p M\} \quad (9.1)$$

st,  $\forall p \in M$ ,  $\omega_p = \omega(p) = \omega(p, v_1, \dots, v_k) : T_p M \rightarrow \mathbb{R}$ , is an alternating tensor. Thus we can also say that  $\omega : M \rightarrow \bigsqcup_{q \in M} \mathcal{A}^k(T_q M)$ , st:  $\omega_p \in \mathcal{A}^k(T_p M)$ .

### 9.1 Space of differential forms

- Let  $\Omega^k(M)$  be defined as follows:

$$\Omega^k(M) = \{\omega | \omega \text{ a smooth differential form of degree } k\} \quad (9.2)$$

We can also define the similar  $\Omega_\delta^K(M)$  as:

$$\Omega_\delta^k(M) = \{\omega | \omega \text{ same as above but not necessarily smooth}\} \quad (9.3)$$

We then have that:  $\Omega^0(M) = C^\infty(M) = \{f : M \rightarrow \mathbb{R} | f \text{ smooth}\}$ .

- $\Omega^K(M)$  is a vector space under point-wise addition/ multiplication with scalars.

### 9.2 Basis $k$ forms

- Recalling that  $e^j(x_1, x_2, \dots, x_n) = x_j \in \mathbb{R}$ , for  $x \in \mathbb{R}^n$ . If we look at the form of  $\psi_I(x)$  in 7.4, and the definition of the wedge product in 8.3 then we can see that we can re-write  $\psi_I$  as:

$$\psi^I = e^{i_1} \wedge e^{i_2} \dots \wedge e^{i_k} \quad (9.4)$$

And we end up denoting this:

$$\psi^I = dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_k} = dx^I \quad (9.5)$$

These  $\psi_I$  are called elementary  $k$  forms (since each  $e^{i_1}$  is a 1-form). It is also worth noticing that:

$$dx^I(v_1, v_2, \dots, v_k) = \det([v_1, v_2 \dots v_k]) \quad (9.6)$$

- We then have that each  $k$  tensor  $\omega(p)$  can be written uniquely in the form:

$$\omega(p) = \sum_{[I]} b_I(p) dx^I(p) \quad (9.7)$$

Where  $[I]$  denotes any increasing sequence, that means no repeating index's, but it does not have to be of length  $k$ . If each  $b_I(p)$  is smooth, then so is  $\omega$ .

### 9.3 Pullback

- Let  $f : M \rightarrow N$  be smooth, we define  $f^* : \Omega^k(N) \rightarrow \Omega^k(M)$  as:

$$(f^*\omega)(p, v_1, \dots, v_k) := \omega(f(p), Df_p v_1, Df_p v_2, \dots, Df_p v_k) \quad (9.8)$$

$f^*$  is a well defined linear map. We may also write  $(f^*\omega)_p \in A^k(T_p M)$ . With these definitions it can be proven that:

- $\text{id}_M^* = \text{id}_{\Omega^*(M)}$
- $(f \circ g)^* = g^* \circ f^*$  For  $L \xleftarrow{f} N \xleftarrow{g} M$  smooth. This map is called a *pullback*.

#### 9.3.1 Smoothness condition

- An element  $\omega \in \Omega_\delta^k(M)$  is smooth if  $\forall p \in M, \exists \alpha : U \rightarrow V \subset M$  a co-ord patch around  $p$  st,  $\alpha^*\omega \in \Omega_\delta^k(U)$  is smooth.

#### 9.3.2 Differential form of wedge

- Let  $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ , we define  $\omega \wedge \eta \in \Omega^{k+l}(M)$  by:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p \quad (9.9)$$

- It can also be shown that if we have  $f : M \rightarrow N$  smooth, then  $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

#### 9.3.3 Conclusion

- $\Omega^\bullet(M) := \bigoplus_{k=0}^\infty \Omega^k(M)$  is a *graded-commutative (anti-commutative) associative* algebra structure, with  $\Omega^0(M) = C^\infty(M)$  (the set of all smooth 1-d functions on  $M$ ).
- It also has that if  $f : M \rightarrow N$ , then  $f^* : \Omega^\bullet(N) \rightarrow \Omega^\bullet(M)$ , preserves the above structure.

It is also worth noting that  $\Omega^k(M) = 0$  for  $k > \dim V$  as then there are more elements in the sequence  $i_1, \dots, i_k$ , then there are dimensions, so  $dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k}$ , must have a repeating index, making it 0 and thus each  $\omega$  is also 0.

## 10 Exterior derivative

- Let  $M$  be a manifold with a boundary,  $\exists$  a unique linear map  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  defined for all  $k \geq 0$  st:
  - If  $f \in C^\infty(M) = \Omega^0(M)$  then  $df_p(v) = Df(p)v$
  - If  $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ , then  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ .
  - $d(d\omega) = 0$ , more over if  $F : M \rightarrow N$  is smooth then  $d(F^*\omega) = F^*d\omega$

### 10.0.1 Exterior Derivative of k-forms

- We know how this derivative acts on 0-forms based on the first property, but how does it act on a  $k$ -form  $\omega$ ? To find this we can just look at our expression of  $k$ -forms in terms of the elementary  $k$ -forms in 9.7. With this expression  $d\omega$  is defined as:

$$\begin{aligned} d\omega &= d \sum_{[I]} b_I dx^I = \sum_{[I]} db_I \wedge dx^I + \sum_{[I]} b_I d^2 x^I \\ &= \sum_{[I]} db_I \wedge dx^I \end{aligned} \tag{10.1}$$

Where we have used the property that  $d^2 = 0$ .

- One can also use this expression to show that  $d^2\omega = 0$  as since  $b_I$  is a 0 form,  $d^2 b_I = \sum_{i,j=1}^n D_i D_j b_I$ , i.e. all second order partial derivatives, but since  $b_I$ , must be a smooth function  $D_i D_j b_I = D_j D_i b_I$ , so since  $d^2\omega = \sum_I \sum_{i,j=1}^n D_i D_j b_I dx^I = \sum_I \sum_{i>j} (D_i D_j b_I - D_j D_i b_I) dx^I = 0$ , as swapping  $dx^{i_i}$  and  $dx^{i_j}$ , picks up a minus sign.

### 10.0.2 Alternate definition

- Alternatively if we have  $\alpha : U \rightarrow M$ , be a patch around  $p$ . we can define  $d\omega$  as:

$$(d\omega)_p := ((\alpha^{-1})^* d_U(\alpha^*\omega))_p \tag{10.2}$$

This is well defined and independent of the choice of  $\alpha$ .

## 10.1 Naturality

- Let  $F : U \rightarrow V$  be smooth,  $\omega \in \Omega^k(V)$ . Then:

$$F^*d\omega = d(F^*\omega) \tag{10.3}$$



## 11 Vector Fields

- A vector field is a smooth function  $X : M \rightarrow TM$ , st.  $X(p) \in T_p M$ . We then define the set of all vector fields on our manifold  $M$ :

$$\mathfrak{X}(M) := \{X : M \rightarrow TM \mid X \text{ is a vector field}\} \quad (11.1)$$

### 11.1 Isomorphisms to differential forms

- We can define the following isomorphisms for  $M \subset \mathbb{R}^d$ :

$$\begin{aligned} h_1 : \mathfrak{X}(M) &\rightarrow \Omega^1(M) \\ \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} &\mapsto \sum_{i=1}^d x^i dx^i \\ h_{n-1} : \mathfrak{X}(M) &\rightarrow \Omega^{d-1}(M) \\ \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{pmatrix} &\mapsto \sum_{i=1}^d x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^d \\ h_n : \mathfrak{X}(M) &\rightarrow \Omega^d(M) \\ u &\mapsto u dx^1 \wedge \cdots \wedge dx^d \end{aligned} \quad (11.2)$$

Note that these isomorphisms may not be natural, i.e.  $h_1(F^*x) = F^*h_1(x)$ , is in general not true.

## 12 Integrating forms

- In this section we will define what it means to integrate a  $k$ -form over a manifold  $M$ . To do this we link our integrals back to  $\mathbb{R}^n$ , where we have well defined integration.

### 12.1 Fubini's Theorem

$$\int f(x^1, \dots, x^n) dx^1 \cdots dx^n = \int_{\mathbb{R}^{n-l}} \left[ \int_{\mathbb{R}^l} f(x^1, \dots, x^n) dx^1 \cdots dx^l \right] dx^{l+1} \cdots dx^n \quad (12.1)$$

### 12.2 Change of Variables

- Let  $F : U_1 \rightarrow U_2$ , ( $U_1 \subset \mathbb{H}^d, U_2 \subset \mathbb{H}^n$ ) a diffeomorphism, then:

$$\int_{\mathbb{R}^n} f(F(x)) |\det DF| dx^1 \cdots dx^n = \int_{\mathbb{R}^n} f(x) dx^1 \cdots dx^n \quad (12.2)$$

#### 12.2.1 Claim

- Change of Variables  $\iff \int F^* \omega = \int \omega$ , if  $|\det DF| = \det DF$ .

### 12.3 Compact Support

- We say  $\omega \in \Omega^k(M)$  has *compact support* if  $\text{supp } \omega := \overline{\{p \in M | \omega_p \neq 0\}}$  is compact. We can then denote  $\Omega_c^k(M)$  denote all the  $k$ -forms with compact supports.

### 12.4 Integral of a d-form

- Let  $M \in \mathbb{H}^d$  ( $M$  open),  $\omega \in \Omega_c^d(M)$ . We define the integral of this  $d$ -form as follows:

$$\int_M \omega = \int_{\mathbb{R}^d} u(x^1, \dots, x^d) dx^1 \cdots dx^d \quad (12.3)$$

Here  $u$  is defined by  $\omega = u dx^1 \wedge \cdots \wedge dx^d$  and is extended by 0 outside of  $M$ .

## 13 Orientations

- In order to determine whether the result of integrating a  $k$ -form, has a  $\pm$  in front of it, we have to define an orientation on the manifold we are integrating over.

### 13.1 Orientation preserving/reversing

- Let  $F : M \rightarrow N$  ( $N, M \subset \mathbb{H}^d$ ), be a diffeomorphism. We call  $F$  *orientation preserving* if  $\det DF(p) > 0$ ,  $\forall p \in M$  and *orientation reversing* if  $\det DF(p) < 0$ ,  $\forall p \in M$ . Note that  $\det DF \neq 0$ , as  $F$  is a diffeomorphism.

### 13.2 Proposition

- Let  $F$  be as above and orientation preserving. Then:

$$\int_M F^* \omega = \int_N \omega, \quad \forall \omega \in \Omega_c^d(N) \quad (13.1)$$

### 13.3 Overlapping Charts

- Let  $M$  be a manifold with a boundary. Let  $\alpha_i : U_i \rightarrow V_i \subset M$  be charts. We say two charts  $\alpha_1$  and  $\alpha_2$  *overlap positively* if  $\alpha_2^{-1} \circ \alpha_1 : \alpha_1^{-1}(V_1 \cap V_2) \rightarrow \alpha_2^{-1}(V_1 \cap V_2)$  is orientation preserving.

### 13.4 Oriented manifold

- An orientation on  $M$  is the choice of collection of charts that pairwise overlap positively and cover  $M$ .

We denote an oriented manifold by  $(M, \{\alpha_i\})$ .

We call a chart  $\beta : U \rightarrow V$ , positive if it overlaps positively with all  $\alpha_i \in \{\alpha_i\}$ . It is easy to see then that.  $\{\alpha_i\} \subset \{\beta\} \iff$  they define the same collection of positive charts.

### 13.5 Reversing orientation

- Set  $\tau : \mathbb{H}^d \rightarrow \mathbb{H}^d$ ,  $(x^1, \dots, x^d) \mapsto (-x^1, \dots, x^d)$ . Given a patch  $\alpha : U \rightarrow M$ , then  $\alpha \circ \tau$  is also a patch with opposite orientation. Usually we denote this by:  $(M, \{\alpha_i \circ \tau\}) = -(M, \{\alpha_i\}) = -M$ .

It's clear to see that if we have  $M$  orientated then either  $\alpha$ , or  $\alpha_i \circ \tau$ , is positive.

### 13.6 Extension of interior orientation

- Let  $M$  be a manifold with a boundary. Suppose  $\overset{\circ}{M} = M \setminus \partial M$  is orientable, then so is  $M$ . Moreover, if  $A = \alpha_i$ , is an orientation on  $\overset{\circ}{M}$ , then  $\exists$ ,  $B = \beta_i$  an orientation on  $M$ , st:  $A \subset B$ .

#### 13.6.1 Corollary

- Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth and 0 a regular value. Then  $f^{-1}([0, \infty]) = M$ , carries a natural orientation.

### 13.7 Induced orientation

- Let  $(M, \{\alpha_i\})$ , be a oriented manifold with a boundary. Then,  $(\partial M, \alpha_i \Big|_{\partial \mathbb{H}^d \cap U_i})$ , is an oriented manifold with what we call a *restricted orientation*.
- The *Induced orientation* on  $\partial M$  is  $(-1)^d$  times the restricted one. This is so that stokes theorem always holds!

### 13.8 Oriented maps

- Let  $f : M \rightarrow N$  be a diffeomorphism and  $(M, \{\alpha_i\}), (N, \{\beta_i\})$ , oriented manifolds. We say  $f$  is orientation preserving if  $\{f \circ \alpha\}$  are positive charts with respect to  $\{\beta_i\}$ .

### 13.9 Positive charts wrt d-forms

- Given  $\omega \in \Omega^d(M)$  on  $M$  a  $d$ -manifold with a boundary. We declare  $\alpha : U \rightarrow M$ , to be positive iff  $\alpha^*\omega \in \Omega^d(U)$ ,  $U \in \mathbb{R}^d$ , st:  $\alpha^*\omega = u dx^1 \wedge \cdots \wedge dx^d$ , with  $u(x) > 0$ ,  $\forall x \in U$ .
- This defines an orientation  $\iff \omega_p \neq 0$ ,  $\forall p \in M$ .

### 13.10 Volume Forms

- $\omega \in \Omega^d(M)$  is called an *volume form* if  $\omega_p \neq 0$ ,  $\forall p \in M$ .

## 14 Stokes Theorem

### 14.1 The integral

- The integral is linear:

$$\int_M \lambda\omega + \eta = \lambda \int_M \omega + \int_M \eta \quad (14.1)$$

If  $-M$ , denotes  $M$ , with the opposite orientation then:

$$\int_{-M} \omega = - \int_M \omega \quad (14.2)$$

### 14.2 Stokes Theorem

- Let  $M$  be an oriented manifold with a boundary, then for  $\omega \in \Omega_c^{d-1}(M)$ :

$$\int_M d\omega = \int_{\partial M} \omega \quad (14.3)$$

#### 14.2.1 Corollary

- If  $M$  has no boundary ( $\partial M = \emptyset$ ), then  $\int_M d\omega = 0$ .

