Temperature from Quantum Field Theory

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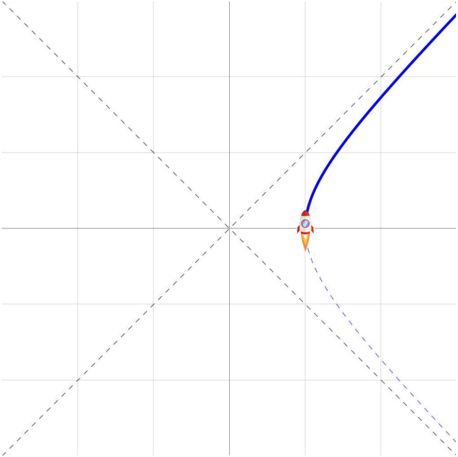
Outline

The Unruh Effect

Rindler Observers



Rindler Observer

Equations of Motion:

$$t(\tau) = \frac{1}{a}\sinh(a\tau)$$
$$x(\tau) = \frac{1}{a}\cosh(a\tau)$$

Rindler observers take a hyperbolic path through spacetime approaching the speed of light. See ACM II Homework 4 for proof.

Rindler Co-ordinates

Relations to x and t

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta), \quad x = \frac{1}{a} e^{a\xi} \cosh(a\eta)$$

- This makes the Minkowski metric $ds^2 = -dt^2 + dx^2$ become:

$$ds^2 = e^{2a\xi} \left[-d\eta^2 + d\xi^2 \right]$$

- Simplest quantum fields: spin 0 massless Klein Gordon particles.
- In Rindler spacetime the Klein Gordon equation $\Box\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$, becomes:

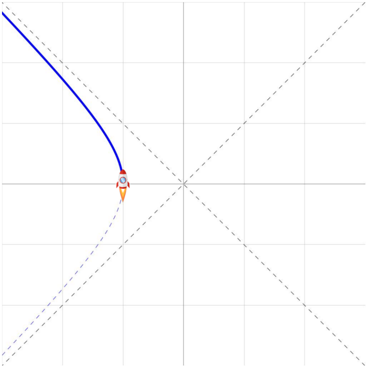
$$\left(-\partial_\eta^2 + \partial_\xi^2 \right) \phi = 0$$

- But wait, have we covered all accelerating observers?

Two Causally Disconnected Regions

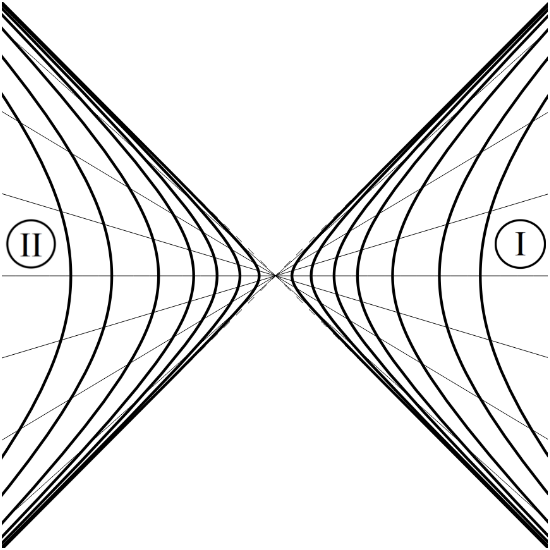
- No! We need more co-ords:

$$t = -\frac{1}{a} e^{a\xi} \sinh(a\eta), \quad x = -\frac{1}{a} e^{a\xi} \cosh(a\eta)$$



Second Rindler Observer

Two Causally Disconnected Regions



Quantising Rindler Spacetime

- Solving the KG equation gives us modes that are $\propto e^{i(-\omega\eta+k\xi)}$. But we need to do this for the two regions:

Two sets of modes

$$g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega_k}} e^{-i\omega_k\eta+ik\xi} & \text{Region I} \\ 0 & \text{Region II} \end{cases}$$

$$g_k^{(2)} = \begin{cases} 0 & \text{Region I} \\ \frac{1}{\sqrt{4\pi\omega_k}} e^{+i\omega_k\eta+ik\xi} & \text{Region II} \end{cases}$$

- This leads to the expansion of the scalar field ϕ:

$$\phi = \int_{-\infty}^{\infty} dk \left[b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*} \right]$$

- Where $[b_k^\dagger, b_{k'}] = (2\pi)\delta(k-k')$.

Extension to Minkowski co-ordinates

- **Goal**: Figure out what the Minkowski vacuum looks like to the Rindler observer.
- Need to express Minkowski modes in terms of Rindler modes.

Rindler modes in terms of x and t

$$\sqrt{4\pi\omega}g_k^{(1)} = a^{\frac{i\omega}{a}}(x-t)^{\frac{i\omega}{a}}$$

- But we need to cover all of Minkowski spacetime? So we need to add:

$$\sqrt{4\pi\omega}g_{-k}^{(2)*} = a^{\frac{i\omega}{a}}(-x+t)^{\frac{i\omega}{a}}$$

- So we can combine:

$$h_k^{(1)} \propto \left[g_k^{(1)} + (-1)^{\frac{i\omega}{a}} g_{-k}^{(2)*}\right]$$

Normalization

- If we normalize these modes we find the constant of proportionality is:

$$A = \frac{1}{\sqrt{1 - e^{-\frac{2\pi\omega}{a}}}}$$

- This means we can expand the scalar field ϕ:

$$\phi = \int_{-\infty}^{\infty} dk \left[c_k^{(1)} h_k^{(1)} + c_k^{(1)\dagger} h_k^{(1)*} + c_k^{(2)} h_k^{(2)} + c_k^{(2)\dagger} h_k^{(2)*} \right]$$

- Where $c_k \left|0_M\right\rangle = 0$.

$$= \int_{-\infty}^{\infty} dkA \left[c_k^{(1)} \left(g_k^{(1)} + e^{-\frac{\pi\omega}{a}} g_{-k}^{(2)*} \right) + c_k^{(1)\dagger} \left(g_k^{(1)*} + e^{-\frac{\pi\omega}{a}} g_k^{(2)} \right) + (1) \leftrightarrow (2) \right]$$

$$= \int_{-\infty}^{\infty} dk \left[b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*} \right]$$

Particles

Relation of b_k and c_k

$$b_k^{(1)} = \frac{1}{\sqrt{1 - e^{-\frac{2\pi\omega}{a}}}} \left(c_k^{(1)} + e^{-\frac{\pi\omega}{a}} c_{-k}^{(2)\dagger} \right)$$

- Meaning the Number operator is:

$$\langle N_k \rangle = \langle 0_M | \, b_k^{(1)\dagger} b_k^{(1)} \, | 0_M \rangle = \frac{e^{-\frac{2\pi\omega}{a}}}{1 - e^{-\frac{2\pi\omega}{a}}} \langle 0_M | \, c_{-k}^{(2)} c_{-k}^{(2)\dagger} \, | 0_M \rangle$$

- So we have:

$$\langle n_k \rangle = \frac{\langle N_k \rangle}{V} = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}$$

Temperature

- This is the occupancy number for a Planck distribution with temperature:

Unruh Temperature

$$T = \frac{a}{2\pi}$$

- So an accelerating observer will see the Minkowski vacuum as a thermal bath with a temperature!
- If we return to SI units and restore the factors we get that this temperature is:

$$T = \frac{\hbar a}{2\pi ck} \simeq 4.055 \times 10^{-21} a \text{ [K]}$$

Hawking Radiation

Black Hole Spacetime

Schwartzchild Metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Hawkings calculation considered comparing vacuum in the far past far from the black hole, to the vacuum in the future close to the black hole.

Particle Creation by Black Holes

S. W. Hawking
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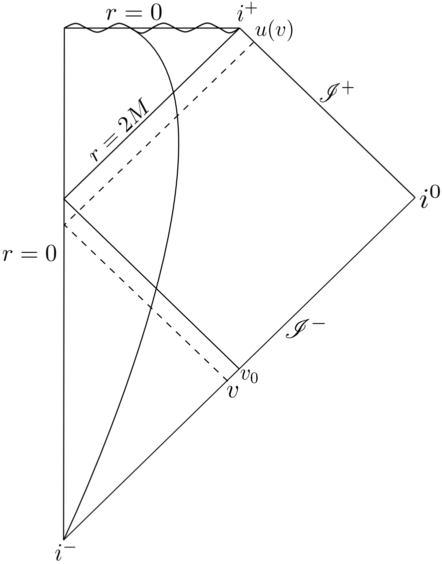
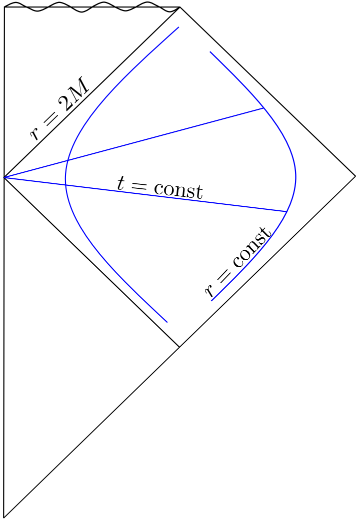
Received April 12, 1975

Abstract. In the classical theory black holes can only absorb and not emit particles. However it is shown that quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature $\frac{\hbar\kappa}{2\pi k} \approx 10^{-6}\left(\frac{M_\odot}{M}\right)$ °K where κ is the surface gravity of the black hole. This thermal emission leads to a slow decrease in the mass of the black hole and to its eventual disappearance: any primordial black hole of mass less than about 10^{15} g would have evaporated by now. Although these quantum effects violate the classical law that the area of the event horizon of a black hole cannot decrease, there remains a Generalized Second Law: $S+\frac{1}{4}A$ never decreases where S is the entropy of matter outside black holes and A is the sum of the surface areas of the event horizons. This shows that gravitational collapse converts the baryons and leptons in the collapsing body into entropy. It is tempting to speculate that this might be the reason why the Universe contains so much entropy per baryon.

Picturing Black Holes



Picturing Black Holes



Horizon Observers

- When we relate the co-ordinates near the black hole to those in far away in the past, we find:

Hawking Radiation

$$\langle n_\omega \rangle = \frac{1}{e^{8\pi M\omega} - 1}$$

- Where the Temperature is now $T = \frac{1}{8\pi M} = \frac{\kappa}{2\pi}$. κ is the *surface gravity*.
- Where does the myth of particle and anti-particle come from?
- The exact same spectrum can be calculated for ingoing particles.
- Both occupation numbers are "entangled" in the sense that their occupation numbers are correleated.

"Consideration of particle emission from black holes would seem to suggest that God not only plays dice, but also sometimes throws them where they cannot be seen."

-*Stephen Hawking*

Thanks for Listening!

Appendix 1

How does the KG equation change in curved spacetime? Recall we change:

$$S = \int \mathcal{L} d^n x \to \int \sqrt{|g|} \mathcal{L} d^n x$$

So the KG is action $\mathcal{L} = \frac{1}{2}\sqrt{|g|}\left(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - m^2\varphi^2\right)$. This has the equations of motion $g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi \equiv \Box\varphi = 0$. We can make the ansatz that the field can be separably written as:

$$\varphi = \frac{1}{r} f(r,t) Y_{\ell m}(\theta, \phi)$$

In the case of the Schwartzchild metric the solution for f is:

$$-\partial_t^2 f + \partial_{r^*}^2 f - \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right) f = 0$$

Both far from the black hole ($r >> 0$) and close to the horizon $r \sim 2GM$, this potential is ~ 0, so f satisfies the KG equation.

Appendix 2

- What do null geodesics look like in this spacetime? Solution define r^*:

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left[-dt^2 + dr^{*2}\right] + r^2 d\Omega_2, \quad r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

- Null geodesics must have $-dt^2 + dr^{*2} = 0 \implies dt = \pm dr^*$.
- These two solutions correspond to in going and out going geodesics:

In-going	Out-going
$v = t + r^*$	$u = t - r^*$

How do we describe observers near the horizon when $(1 - \frac{2GM}{r}) \to 0$?.

Kruskal co-ordinates

$$U = -e^{-u/4GM}, \quad V = e^{v/4GM}$$