Lecture 1

1 Introduction

The contents of this lecture will be to introduce gravity, compare it to classical electromagnetism to ease into the field theory formulation, then explain how the differences lead to large-scale physical effects. We should also mention the postulates of GR. Also want to discuss the weakness of gravity and why that leads to its long-scale size, no characteristic length scale like weak force etc....

Resources: Maybe link to my notes on CFT for the EM section. Also should recommend the books Jackson and Griffiths, Brian Dolan's book as well as Hartle's?

"Zierler: Was general relativity considered popular or interesting at Princeton at the time that you were a graduate student?

Witten: Well, I was certainly interested in it. I learned general relativity in a very exciting period of about ten days, from the book of Steve Weinberg. I mean, I tried to learn more from the book of Misner, Thorne, and Wheeler, and I did learn more from it, but my opinion of the book was what it remains now, which is that it's got a lot of great stuff in it, but it's a little bit hard to use it to learn systematically. The book I found useful for studying systematically was by Steve Weinberg."

From a great article by Peter Woit.

2 Gravity and Electromagnetism

We start by recalling Newtonian gravity. We write down the familiar force law:

$$F = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} \tag{1}$$

From the get-go this theory has problems when we try to combine it with special relativity. Newton's forces are instantaneous, no matter the distance between the two masses. We know that information cannot travel faster than the speed of light. Hence, this very nice theory must be nothing more than an approximation of a deeper theory.

Looking at this first equation, we cannot help but note its similarity to Coulomb's Law:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

Both are inverse-square laws, both involving two charges of sorts. So why is our experience of the force of gravity so different from that of electromagnetism? There are two reasons for this:

- The force of gravity is extremely weak: $\frac{F_{\text{grav}}}{F_{\text{elec}}} \sim 10^-36$ for two protons separated at any

distance.

- The minus sign in front of the first equation, along with the fact that there are no negative charges in

gravity, means that gravity has no shielding and will thus be accumulative, whereas EM can have screening effects. a These two facts explain why eventually gravity comes to dominate in terms of length scales. It also explains why these length scales are so large.

Note also that the $\frac{1}{r^2}$ potential is long-range, compared to the weak interaction or strong force between nucleons, which have massive force carriers and thus a Yukawa potential and a characteristic scale that makes them fall off far faster than inverse-square-law forces.

In talking about GR, we are concerned with gravitational phenomena that are relativistic. Our talk of scales may make you wonder: what is the scale at which gravity becomes relativistic? Is it a length scale, a certain speed like in special relativity? Well, to properly find this out, we will need to fully analyze the theory. But there is a naive approach that gets you close to the right answer. We simply take all the relevant constants and combine them to get a scale. It's most convenient for reasons we'll see later for this to be a length scale.

The relevant constants for a relativistic theory of gravity, which necessarily involves a massive object M, are M itself, G (Newton's constant), and c (the speed of light). You can check that the only combination of these that has units of length is:

$$r \sim \frac{GM}{2c^2}$$

This is indeed the Schwarzschild Radius! We will eventually discuss this in more detail. You may have heard of simple ways of deriving this by imposing that the escape velocity of a body of mass M becomes the speed of light. That method is lucky to get the correct answer — there's actually no physical reason it should. The real Schwarzschild radius has less to do with escape velocity than you might think. But we'll discuss this more later.

This characteristic scale means that if we have an object of mass M that is smaller than or near the size of this scale, then general relativistic effects will be prevalent. It matters that the object be small enough, not big, because the gravitational force at a radius r depends on the total amount of matter enclosed in a sphere of radius r. So the denser an object is, the more relativistic it will become. This scale exists for every object, and you can calculate whether the object's actual radius is bigger than its Schwarzschild radius. You will find that naturally most objects are far bigger than this radius.

Some examples:

- Earth: $R_\oplus \sim 10^6 m$, but $R_s \sim 10^{-3} m$
- Proton: $R_p \sim 10^{\{-15\}} m$, but $R_s \sim 10^{-54} m$

Where this becomes insteresting is when we consider elementary particles like electrons that are meant to be point like and should have no radius.

3 Gravity as a Field

The first step toward general relativity is to treat gravity as a field. Michael Faraday was the first to abstract Coulomb's law of charges to that of fields. He said that a charge q_1 generates what came to be called an electric field:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

Then the force on a particle of charge q_2 is:

$$F = q_2 E$$

The electromagnetic force is conservative, which means:

$$\nabla \times F = 0 \Rightarrow \nabla \times E = 0$$

This may look familiar — it's one of Maxwell's equations for static EM fields. This means we can write E as (minus) the gradient of a scalar potential ϕ:

$$E = -\nabla\phi$$

And Gauss's Law gives:

$$\nabla \cdot E = -\nabla^2\phi = \frac{\rho}{\epsilon_0}$$

We can do the exact same for gravity! We can think of a mass M creating a gravitational field g:

$$g = -\frac{GM}{r^2}\hat{r}$$

Then a mass m experiences a force:

$$F = mg$$

The gravitational force is also conservative, so we can use the same trick of writing:

$$g = -\nabla\phi$$

The gravitational Gauss's law becomes:

$$\nabla \cdot g = -\nabla\phi = -4\pi G\rho \tag{2}$$

4 The Equivalence Principle

Einstein and many Philosophers of physics like to categorize physical theories into two categories

1. Constructive theories
2. Principle theories

The former is based on models of systems which are often based on empirical data (Electromagnetism, Thermodynamics), where as the latter is based entirely on principles, with everything else being derived from the principles. The later are hence considered more robust.

Apart from the sign in 2 , gravity and Electromagnetism seem to have the same foundations. So what makes gravity so special, why do we need such a radically different theory such as GR? Forces cause things to accelerate, and it's here that gravity and electromagnetism differ. Newton's second law tells us that the force on an object is proportional to the acceleration that object undergoes, with the constant being a measure of the object's resistance to move, its inertia. This is the inertial mass.

Why does this make gravity different? In Newtonian physics, we assume that the inertial mass is exactly the same as gravitational mass. This means that when we equate the force equations:

$$F = mg$$

and

$$F = ma$$

we get:

$$a = g$$

This has big consequences: it means that two objects of different mass will accelerate at the same rate. You may have seen David Scott dropping a hammer and a feather on the Moon. This same idea extends to a postulate of GR — the equivalence principle, though it takes several forms:

- The Weak Equivalence Principle states that objects bound by non-gravitational forces fall with exactly the same acceleration.
- The Strong Equivalence Principle extends this to bodies bound by gravitational forces, like planets. This is a much stronger condition and is assumed in GR, though GR is thought to be the only theory of gravity that satisfies it.

The French satellite MICROSCOPE has performed the most precise test of the equivalence principle with a sensitivity of 10^{-15}.

What the equivalence principle tells us that gravitational acceleration is indistinguishable from regular acceleration. Einstein took this idea as an inspiration to formulate gravity in a geometrical way such that there is no gravitational force, just curvature. Having formulated special relativity where space and time are intimately linked. Einstein realized this would not just be a curvature of regular 3 dimensional space, but a curvature of 4 dimensional space time.

5 Discussing Geometry

To summarize we have concluded that gravity, due to the unique feature of the charge of the gravitational force (mass) being the same as inertial mass, we may reformulate the way we think about gravity as not a force but as a curvature of spacetime itself.

This way of thinking abouit gravity tells us that what we once thought to be paths that were being curved by the force of gravity (such as a ball thrown up in the air) are actually the new “straight lines”. Obviously they are not straight to us, but what I mean by this is that these are the paths traveled by the particles which experience no forces. These are known as geodesics and we will discuss them in far more detail later.

The question then is how can we describe curved space? There are many ways of doing this, but when Einstein came to this problem in the early 1900s he was lucky enough to find that Mathematicians had been thinking about this for a while. The key to finding the most useful approach is to think about what is the most impotent for your theory.

For example you may have have seen that surfaces can be classified with different curvature based on what is the sum of the inside angles of a triangle on the surface of the curved. For us who think of throwing balls up in the air what is the impotent feature that distinguishes flat space from curved space? What mathematicians have found is that a curved spaces can be classified by reducing all geometry to the specification of the distance between nearby points.

That is that We can do this cleverly by finding the distance between infinitesimally close points, then integration can be used to find the length of paths between two finitely space points. We can call curves that give the shortest distance between two points, straight lines. This reduces to the straight lines we are familiar with in the euclidean case. It makes sense in away that specifying the geometry at every infinitesimal point will with a bit of work tell us the macroscopic geometry.

This area of mathematics is called differential geometry. I recommend having a very good basis in this if you want to study GR seriously. We will not delve into the details too much here as much of the physics can be discussed with out all the formalism.