Lecture 2

In this lecture we introduce special relativity and the concept of metric spaces.

Resources: Tong's Special Relativity notes, Andrew Dotson's series on tensors, Goldstein's mechanics, and maybe Leonard Susskind's book.

6 Metrics

We start with flat space. The distance between points is given by Pythagoras' theorem:

$$\Delta s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

Taking the limit as the points get infinitesimally close:

$$ds^2 = dx^2 + dy^2 + dz^2$$

This is known as the line element. In linear algebra terms, with a vector (dx_1, dx_2, dx_3), the line element becomes:

$$ds^2 = (dx^1, dx^2, dx^3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Or using the Kronecker delta:

$$ds^2 = \sum_{\alpha,\beta=1}^{d} \delta_{\alpha\beta} dx^\alpha dx^\beta = \delta_{\alpha\beta} dx^\alpha dx^\beta$$

Some notation introduced by Einstein is to drop summation symbol here and use the fact that there are upper and lower indices to imply that the indices are being summed over. We can then talk about the objects $\delta_{\alpha\beta}$ in isolation with out reference to what co-ordinates I am talking about. Since the postulates of special relativity (and hence GR) assume that the laws of Physics do not change if we make a change of co-ordinates,. We can discuss physics generally with these Tensors, and can at any time pick the basis system we want to use to calculate some physical results that relate to a specific way of looking at a system (i.e. specific co-ordinates).

This $\delta_{\alpha\beta}$ metric is quite simple, but we will see that for curved spaces it becomes more complicated. For example, in spherical coordinates:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

One last thing to notice is that these metrics have certain symmetries which we can actually use to define them. In flat space, space looks the same no matter which way you rotate and this should be reflected in the metric. This is very easy to see in the case of the spherical metric, we simply just change φ by a constant and it is the same, but even more subtly in the Cartesian metric, we can apply a rotation matrix for a rotation about the z axis:

$$\boldsymbol{x}' = R\boldsymbol{x} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And true to the symmetry we expect this to leave the metric unchanged (i.e. $\boldsymbol{x}'^T \mathbb{1} \boldsymbol{x}' = \mathbb{1}$):

$$ds^2 = (dx^1, dx^2, dx^3)\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}dx^1\\dx^2\\dx^3\end{pmatrix}$$
$$= (dx^1, dx^2, dx^3)\begin{pmatrix}\cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi & 0\\ \cos\phi\sin\phi - \sin\phi\cos\phi & \sin^2\phi + \cos^2\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}dx^1\\dx^2\\dx^3\end{pmatrix}$$

In GR as I mentioned before we are concerned with the curvature of not 3d space, but 4d, including time, hence the metric will not be as simple as it is here.

7 Special Relativity

Special relativity is based on two postulates, from which everything else follows:

1. The form-invariance of physical laws in all inertial frames.
2. The constancy of the speed of light in vacuum.

I unfortunately do not have time to cover this in detail but i will point you to David Tongs notes as a good resource for learning, or if you are completely new I found Lenard Susskinds Theoretical Minimum to be a great introduction.

Anyway what I will assume you are familiar with is the fact that the second postulate gives rise to the following transformations rules:

$$t' = \gamma\left(t - v\frac{x}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

We can then cleanly write this in matrix form, by grouping the space and time components into a single vector that we call a four vector. Then what we call a Lorentz transformation takes the form :

$$\begin{pmatrix}ct'\\x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\frac{v}{c} & 0 & 0\\ -\gamma\frac{v}{c} & \gamma & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}ct\\x\\y\\z\end{pmatrix} \tag{3}$$

The inverse of this must multiply by this matrix to give the identity (as a boost in one direction followed by a boost in the opposite direction should result in nothing happening), but it is just a two by two matrix so we can simply write down that:

$$\Lambda^{-1}\Lambda = \frac{1}{\det\Lambda}\begin{pmatrix}\gamma & \gamma\frac{v}{c} & 0 & 0\\ \gamma\frac{v}{c} & \gamma & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\gamma & -\gamma\frac{v}{c} & 0 & 0\\ -\gamma\frac{v}{c} & \gamma & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix} = \frac{1}{\det\Lambda}\begin{pmatrix}\gamma^2\left(1-\frac{v^2}{c^2}\right) & 0 & 0 & 0\\ 0 & \gamma^2\left(1-\frac{v^2}{c^2}\right) & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}$$

So nicely we see that $\det\Lambda = 1$. This allows us to parameterise this matrix in a very similar way to the rotation matrix:

$$\Lambda = \begin{pmatrix}\cosh(\psi) & \sinh(\psi) & 0 & 0\\ \sinh(\psi) & \cosh(\psi) & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}$$

One can compare this to the other matrices to get the exact paramatrization. The question is then what metric is left is left invariant under these transformations? It is not just a diagonal 4×4 matrix as that does not give us the sign difference between the sinh and cosh:

$$\begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh^2\psi + \sinh^2\psi & 0 & 0 & 0 \\ 0 & \cosh^2\psi + \sinh^2\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But we can easily see a way that will fix this, we simply change the sign of the first component:

$$\begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cosh^2\psi + \sinh^2\psi & 0 & 0 & 0 \\ 0 & \cosh^2\psi - \sinh^2\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which is exactly what was needed. So what this means, is that the only metric that respects the symmetry of a Lorentz transformation is the one we just derived. We can then go ahead and see what line element this corresponds to:

$$ds^2 = (dx^0, dx^1, dx^2, dx^3) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \tag{4}$$
$$\Rightarrow ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

8 Proper Time

So we have shown that we can can combine space and time into a single 4-vector x^μ. This object marks the location of events. The "magnitude" of this vector takes a very different form to that of regular 3-d vectors. Since the metric is no longer euclidean, the magnitude is not the sum of the square of the components, but rather takes the same form as the infinitesimal line element with the time component picking up a minus sign:

$$x^2 := (x^0, x^1, x^2, x^3) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

Why do we do it this way? Well this ensures that this quantity that we have measured is invariant under Lorentz transformations!

We can ask ourselves the question, what is this conserved quantity? Well since it is conserved we can "Boost" to a frame that makes it make the most sense.

Begin drawing Minkowski diagram here

If we think of any two points, we can always boost to a frame such that in the new frame the spacetime event is either directly ahead in time such that $x^2 = -(x^0)^2$ or directly along in space such that $x^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. This allows us to interprete this conserved quantity as either the proper time in the frame of an observer traveling between these points. If the points are spacelike seperated this is simply be how far a way an observer boosted to the velocity would see this point as being.