Lecture 5

In this lecture we will discuss parallel transport and the geodesics of point particles.

15 Parallel Transport

Having drastically altered the ideas of Newtonian physics to encorperate curved spacetime, we now need to try rebuild the laws of physics that objects obey in this new theory. In Newtonian mechanics, Newtons first law stated that a body either remains at rest or travels in a straight line unless acted on some force. How can we rephrase this in the new language of curved space that we have created. Clearly if gravity is no longer a force and simply a manifestation of the curvature of spacetime, then this law no longer holds for gravity. Instead objects acted upon by no fores will sometimes take curved trajectories. How to do we know what trajectories they will follow? To answer this question we will need to develop the concept of parallel transport.

In flat space "a body traveling in a straight line" translates to the velocity vector of a trajectory is constant, in that it does not change through out space. We can sum this up as along the direction of motion of the trajectory (which we can achieve by dotting with the tangent vector V^μ) the vector field should not change:

$$V^\mu \partial_\mu V^\nu = 0$$

Visually the vector that goes in a straight line is always parallel to itself if we compare it at any two points, hence another way of thinking about this is to say that the vector parallel transports itself along the straight line. But how does this work in curved spacetime, where the bases are no longer constant.

If we look back at equation 15 it breaks down what happens to a vector as you move along an axis in space in a remarkably simple way. As you go from one point to the next in space or spacetime, the vector changes for two reasons. The first is the simple intrinsic change in the vector V as you travel along some path. This occurs in any non-trivial vector field and its contribution to the co-variant derivative is the first term in 15, given by the partial derivative of V^μ. The second type of change is a change in the basis vectors that the vector is expanded over. Equation 14 tells us how the bases vectors change from point to point in curved spacetime and this contribution is seen exactly in the second term of 15.

So what we see is that in curved space is that a vector parallel transporting itself can no longer travel in a straight line due to the movement of the bases. So we need a new condition other then $V^\mu \partial_\mu V^\nu = 0$. This condition must hold in all co-ordinates (hence be a tensor), and must reflect the $V^\mu \partial_\mu V^\nu$ condition + the offset due to the bases. This is exactly given by the dotting of V^μ with the co-variant derivative. So we say a vector parallel transports itself if:

$$V^\mu \nabla_\mu V^\nu = 0 \tag{18}$$

Sometimes this is also written as $\nabla_V V^\nu = 0$. Using 15 we can expand out this equation. Note that we are often concerned with the trajectory $x^\mu(\lambda)$ to which our vector $V^\mu = \frac{dx^\mu}{d\lambda}$ is tangent to. So in terms of the trajectory the parallel transport equation 18 says:

$$\frac{dx^\mu}{d\lambda}\left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma\right) = 0$$

$$= \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma} \frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{19}$$

Where we have recognized the chain rule on the first term. A nice sanity check is that if we are in flat spacetime then all the $\Gamma^\nu_{\mu\sigma} = 0$ and this reduces to $\frac{d^2 x^\mu}{d\lambda^2} = 0$, which is the equation for a straight line.

Note also that parallel transport is not limited to vectors transporting themselves, we can also have that a vector W^μ is transported along the trajectory $x^\mu(\lambda)$ if:

$$V^\mu \nabla_\mu W^\mu = 0 \tag{20}$$

Where $V^\mu = \frac{dx^\mu}{d\lambda}$ is the tangent vector to the trajectory. In this sense the vector W^μ is transported along the trajectory in a way that keeps it "fixed" in the same direction the most.

16 Action for point particles

We have derived an equation that mimics Newtons first Law, this is very useful and allows us to calculate many things. But just like in classical mechanics, the existence of a law like this is only as useful as it is solvable. In more complicated spacetimes this differential equation may become very coupled and complicated, and we may need a different formalism if we want to solve these problems.

An alternate way of looking at the problem is the Lagrangian formulation of mechanics. Here we say that the trajectory that the particle actually takes, maximizes (or minimizes) some quantity S which we call the action. Specifically we then say there is a function called the Lagrangian L that takes values for each possible postilion q and velocity $\dot{q}$ of the particle and that the action is just the accumulation of this quantity along the trajectory traveled. It is convenient to parameterize the trajectories with time such that:

$$S = \int L(q, \dot{q}) dt$$

The purpose of having a quantity that depends on the q's and $\dot{q}$'s along the trajectory is that it allows us to tell apart the different trajectories and find the one which extremizes the overall action by methods of calculus of variations. The result of this, is that the Lagrangian which does extremize the action must obey the following second order differential equation, known as the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{21}$$

In classical mechanics for example, for single particles with mass m, we can see that if we have the Lagrangian be the kinetic energy $T = \frac{1}{2}m\dot{q}^2$ minus the potential $U(q)$, so $L = \frac{1}{2}m\dot{q}^2 - U(q)$. Then the Euler Lagrange equation 21 reads:

$$m\frac{d\dot{q}}{dt} = -\frac{\partial U}{\partial q}$$

If we recall that $-\frac{\partial U}{\partial q}$ is equal to the force experienced by the mass m, then we can see that this particular Lagrangian recovered Newton's third Law, and hence most of classical mechanics.

When we want to write down the action for other situations such as here in General relativity it is difficult to know where to start. But if we notice a few things it can be made easier.

1. The action is a scalar quantity, it doesn't have any indices and should be Lorentz invariant. This means it must be made from the contraction of indices as we saw in lecture 3.
2. Reparametrization invariance, the parametrization used to describe the trajectory should not be unique other wise that would pick out a preferred time and hence a preferred frame.

We can then begin to think about the setup. We are considering a particle in space traveling from a event y to an event x in some arbitrary curved spacetime described by some metric $g_{\mu\nu}$. We then want to list the quantities that we can use to build our action. In terms of "vector" quantities we have the position vectors x_1^μ and x_2^μ, or more relevantly their difference $x_2^\mu - x_1^\mu$. The only scalars we have are the mass m of the particle and maybe the constants c and G. For simplicity from here on we can let the point x_1^μ be the origin. Then the only quantity we an make out of contractions is $x_2^\nu g_{\mu\nu} x_2^\mu = (x_2)_\mu (x_2)^\mu = \tau^2$, which you may recall from lecture 2 is the proper time of the particle.

But hold on a minuet, we have accidentally commited a faux pas here. When we wrote $(x_1^\mu - x_2^\mu)$ we implicitly compared two position vectors at different points. This may seem fine as they are position vectors not proper vectors in the tangent space, but the main issue arises when we contract $(x_1^\mu - x_2^\mu)$ with the metric. The metric is a function of x so at which point do we evaluate it? x_1? x_2? There is no preferred choice. The point is we cannot compare finitely separated points using the metric, only infinitesimally close points.

Luckily this is inline with the concept of the Lagrangian. We can pick some We can pick some λ to parameterize the trajectory, and then integrate along this to obtain the action, while only comparing local points. Having introduced a new variable, we unlock new possible building blocks in $\frac{dx^\mu}{d\lambda}$ and possibly higher derivatives. If our integral takes the form $\int(\cdots)d\lambda$, then we can clearly see the only Lorentz invariant and preserves reparametrization is:

$$S \propto \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

The reparametrization invariance comes from the fact that if we change from λ to some some other parameter α then from the chain rule $\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \to \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\alpha}\frac{d\alpha}{d\lambda}\frac{dx^\nu}{d\alpha}\frac{d\alpha}{d\lambda}}\frac{d\lambda}{d\alpha}d\alpha = \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\alpha}\frac{dx^\nu}{d\alpha}}d\alpha$. You should convince your self that this is the only possible combination that satisfies the two conditions. The last thing to do is to figure out the units out of our remaining blocks m, c, G. The current quantity has units length, which means we need to multiply it by a mass and a velocity scale, hence the nice clean choice is:

$$S = -mc\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

We didn't involve G as in the flat space limit $g_{\mu\nu} \to \eta_{\mu\nu}$ and hence it would not make sense to a have prefactor involving G for flat space. Lastly we can notice that using the line element we can write this very cleanly as:

$$S = -mc\int ds$$

The minus sign and this choice of constants prefactor can be shown to reduce to the $L = \frac{1}{2}mv^2$ in the flat space and non relativistic ($c \to \infty$) limit. Physically this result means that the trajectory that particles follow is the one that extremizes the distance (proper time) between two events. These types of trajectories are known as **geodesics**.

One may then ask is the proper time maximized or minimized on these trajectories? Tot which the following argument answers nicely. Let us guess that the proper time is minimized. But then we can consider the path of a light ray that zigs and zags to follow the trajectory of the particle. Light rays have (in the limit as one approaches the speed of light) 0 proper time along their trajectories. Hence, these trajectories cannot minimize proper time as they are always infinitesimally close to a path that has less proper time. Hence we can only conclude that these geodesics maximize proper time. This has some fun and powerful consequences. For example we can use it to resolve the twin paradox! The twin that stays at home and does not go off accelerating in a ship follows a geodesic and hence must be the older twin when they return!

17 Eom from the Action

Now that we have a suitable action principle that uniquely satisfies our requirements we want to put it to the test and see if it reproduces the parallel transport equation 19 above. The procedure we need to carry out is to apply the Euler Lagrange equations 21 to this action. However, with all the indices and the metric now being a function of x, this can get messy very fast. To make life easier for ourselves we can notice the following. What we are doing in using the Euler Lagrange equations is extremizing the action by setting $\delta S = \int \delta L d\lambda = 0$. But what if we defined the quantity $G = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ such that $L = \sqrt{G}$. Since the variation δ is just like a derivative but for functionals, we have that $\delta L = \delta\left(\sqrt{G}\right) = \frac{1}{2}\frac{\delta G}{\sqrt{G}}$. But quite nicely, this implies that if $\delta L = 0 \Rightarrow \delta G = 0$. This means inseatd of using our complicated action with a square root we can simply use:

$$S' = \int g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda$$

With this new action we can now apply the equations of motion 21 for each component:

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\partial_\alpha g_{\mu\nu} = \frac{d}{d\lambda}\left[g_{\mu\nu}\frac{\partial}{\partial \dot{x}^\alpha}(\dot{x}^\mu \dot{x}^\nu)\right] = \frac{d}{d\lambda}\left[2g_{\alpha\mu}\frac{dx^\mu}{d\lambda}\right]$$

Where we have used the fact that $\frac{dx^\mu}{d\lambda}$ must be independent of x and have denoted $\dot{x}^\alpha = \frac{dx^\alpha}{d\lambda}$. When we take the derivative of $g_{\mu\nu}$ wrt λ we can use the chain rule to write $\frac{dg_{\alpha\mu}}{d\lambda} = \frac{\partial g_{\alpha\mu}}{\partial x^\gamma}\frac{dx^\gamma}{d\lambda}$. This means:

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\partial_\alpha g_{\mu\nu} = 2\partial_\gamma g_{\alpha\mu}\frac{dx^\gamma}{d\lambda}\frac{dx^\mu}{d\lambda} + 2g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2}$$

We can then relabel indices on this middle terms such that $2\partial_\gamma g_{\alpha\mu}\frac{dx^\gamma}{d\lambda}\frac{dx^\mu}{d\lambda} = \partial_\gamma g_{\mu\alpha}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} + \partial_\gamma g_{\alpha\mu}\frac{dx^\gamma}{d\lambda}\frac{dx^\mu}{d\lambda}$. Also changing the ν label on the HS to γ, we are just left with:

$$\left[\partial_\gamma g_{\mu\alpha} + \partial_\gamma g_{\alpha\mu} - \partial_\alpha g_{\mu\gamma}\right]\frac{dx^\gamma}{d\lambda}\frac{dx^\mu}{d\lambda} + 2g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2} = 0$$

$$\Rightarrow \frac{d^2x^\nu}{d\lambda^2} + \frac{1}{2}g^{\nu\alpha}\left[\partial_\gamma g_{\mu\alpha} + \partial_\gamma g_{\alpha\mu} - \partial_\alpha g_{\mu\gamma}\right]\frac{dx^\gamma}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$$

We can then recognize that this equation is the exact same as the parallel transport 19! We can actually say something stronger. We can say that the choice of metric compatibility is equivalent to requiring that the equation of motion that our particles follow is the parallel transport equation.