Lecture 6

18 Defining curvature

We have discussed various things so far that help us describe and understand spacetimes that are curved. But, we have not quite developed a criteria for saying, given some metric describing a space, what makes us say whether it is curved or not. We may think it is just if the co-ordinate system we are in has some non-zero Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$. But, we have seen for example spherical co-ordinates that describe flat space, despite having non-zero $\Gamma^{\alpha}_{\mu\nu}$, due to the fact that they are curvilinear co-ordinates (the basis vectors are not orthogonal). So we see that the Christoffel symbols are not an indicator as to whether the basis is changing due to the co-ordinates or some intrinsic curvature. We need something better.

A better guess would be to say that a spacetime is flat if there exist a frame in which the Christoffel symbols are all zero at *every* point in spacetime. This is a useful statement but it is not obvious what it means mathematically. We would like to use this definition to write down a mathematical equation that gives us a condition for the curvature of a space. If a spacetime satisfies it then it flat otherwise it is curved. Naturally to have this hold in all frames we want to make this a tensor equation.

If there exists a co-ordinate system where $\Gamma^{\alpha}_{\mu\nu} = 0$, then we know that the co-variant derivative of vectors reduces to the partial derivative. That is that $\nabla_\mu V^\nu = \partial_\mu V^\nu$ (where really we mean $\left(\nabla_\mu V\right)^\nu = \left(\partial_\mu V\right)^\nu = \partial_\mu V^\nu$). One thing we know about partial derivatives is that they commute. This means for any function f, $\partial_i \partial_j f = \partial_j \partial_i f$. But because the co-variant derivative reduces to the partial derivative, we can say:

$$\partial_\mu \partial_\nu V^\alpha - \partial_\nu \partial_\mu V^\alpha = 0 \Rightarrow \nabla_\mu \nabla_\nu V^\alpha - \nabla_\nu \nabla_\mu V^\alpha = \left[\nabla_\mu, \nabla_\nu\right] V^\alpha = 0$$

And as mentioned since this is a tensor equation, transforming it just amounts to multiplying it by a factor of partial derivatives. Hence we can say, that in order for a spacetime to be flat, this equation must be true in any frame, not just the one with vanishing Christoffel symbols.

Let us see if we can get some physical intuition for what is going on here. We can think of the action of the co-variant derivative on a vector $\nabla_\mu V^\nu$ as the parallel transport of the vector V^ν along the direction of the μ'th basis vector. This is because $\nabla_\mu V^\nu$ looks very like our parallel transport equation 21. Then we can think of applying two co-variant derivatives as parallel transporting one way and then the other, i.e. in the μ'th direction and then the ν'th. When we reverse the sign in front of these it results in transporting in the opposite direction. Hence, the commutator amounts to the parallel transporting of a vector around a parallelogram[3], see Figure 1.

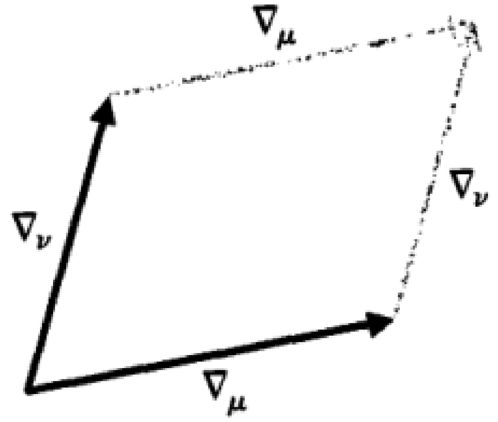

Figure 1: Visualization of the commutator

So what the mathematics tells us is that if we infinitesimally parallel transport a vector around this parallelogram and the resulting vector is exactly the same, i.e. un-rotated, then the spacetime is flat **at**

[3]Note that because these are basis vectors we are always brought back to the same point, though this is not true for general parallel transport.

that point. For the entire spacetime to be flat this needs to hold everywhere. This is great! It gives us a way of distinguishing genuinely curved spacetime from flat ones.

I would now like to look at an example to see this rotation effect more clearly, as the parallelogram is not very illustrative. Consider the sphere in Figure 2, here we parallel transport some vector, starting at the north pole; down to the equator; across the equator; and finally back to the north pole[4]. From this we can clearly see how the vector is rotated.

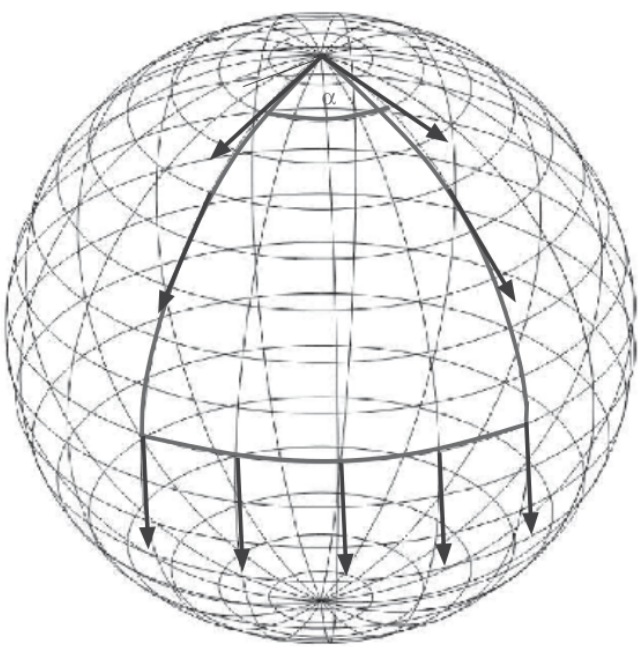

Figure 2: Parallel Transport around a sphere

19 The Curvature Tensor

This method of mathematically defining flat spacetime can actually be used to help us understand curved spacetimes more. Since the commutator of co-variant derivatives applied to a vector must be $\left[\nabla_\mu, \nabla_\nu\right]V^\alpha = 0$ at a point to be flat, that then means if anything non-zero was to show up on the other side of this equation, it would be an indicator of curvature at this point. So what we can do is actually go though the painstaking calculation of this commutator and what ever we have left must be zero for flat spaces and non-zero for curved. We can do this calculation with the help of the definition of the co-variant derivative of a vector defined in 16.

$$\left[\nabla_\mu, \nabla_\nu\right]V^\alpha = \nabla_\mu\nabla_\nu V^\alpha - \nabla_\nu\nabla_\mu V^\alpha$$
$$= \underbrace{\partial_\mu(\nabla_\nu V^\alpha)}_{①} + \underbrace{\Gamma^\alpha_{\mu\gamma}\nabla_\nu V^\gamma - \Gamma^\gamma_{\mu\nu}\nabla_\gamma V^\alpha}_{②} - \underbrace{\partial_\nu\left(\nabla_\mu V^\alpha\right)}_{③} - \underbrace{\Gamma^\alpha_{\nu\gamma}\nabla_\mu V^\gamma + \Gamma^\gamma_{\nu\mu}\nabla_\gamma V^\alpha}_{④}$$

Once again we then have to expand the co-variant derivatives, so this is going to get very messy, but bare with me, we will do it step by step.

$$① - ③ = \partial_\mu\partial_\nu V^\alpha + \left(\partial_\mu\Gamma_\nu\gamma\right)^\alpha)V^\gamma + \Gamma^\alpha_{\nu\gamma}\left(\partial_\mu V^\gamma\right) - \partial_\nu\partial_\mu V^\alpha - \left(\partial_\nu\Gamma^\alpha_{\mu\gamma}\right)V^\gamma + \Gamma^\alpha_{\mu\gamma}(\partial_\nu V^\gamma)$$
$$= \left(\partial_\mu\Gamma^\alpha_{\nu\gamma} - \partial_\nu\Gamma^\alpha_{\mu\gamma}\right)V^\gamma + \Gamma^\alpha_{\nu\gamma}\left(\partial_\mu V^\gamma\right) - \Gamma^\alpha_{\mu\gamma}(\partial_\nu V^\gamma)$$

Also:

$$④ + ② = \Gamma^\alpha_{\mu\gamma}(\partial_\nu V^\gamma + \Gamma^\gamma_{\nu\sigma}V^\sigma) - \Gamma^\gamma_{\mu\nu}\left(\partial_\gamma V^\alpha + \Gamma^\alpha_{\gamma\sigma}V^\sigma\right) + \Gamma^\alpha_{\nu\gamma}\left(\partial_\mu V^\gamma + \Gamma^\gamma_{\mu\sigma}V^\sigma\right) + \Gamma^\gamma_{\nu\mu}\left(\partial_\gamma V^\alpha + \Gamma^\alpha_{\gamma\sigma}V^\sigma\right)$$
$$= \Gamma^\alpha_{\mu\gamma}\partial_\nu V^\gamma - \Gamma^\alpha_{\nu\gamma}\partial_\mu V^\gamma + \left(\Gamma_{\mu\gamma}\Gamma^\gamma_{\nu\sigma} - \Gamma_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\right)V^\sigma$$

[4]Note that the last side of the parallelogram is hidden in the north pole

Where we are using the property that $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. This means all together we can write:

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] V^\alpha = ① - ③ + ④ + ② &= (\partial_\mu \Gamma^\alpha_{\nu\gamma} - \partial_\nu \Gamma^\alpha_{\mu\gamma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\gamma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\gamma})V^\gamma \\ &= R^\alpha{}_{\gamma\mu\nu} V^\gamma \end{aligned} \tag{23}$$

This rank $(1,3)$ tensor is know as the **Riemann Curvature Tensor**. This is a nice result as the tensor $R^\alpha{}_{\gamma\mu\nu}$ is independent of the vector V^α, hence we can say that this is a good indicator of the curvature at some point. We also have the condition, that for a space to be flat this curvature tensor must vanish everywhere.

It also turns out that this tensor is in fact the only tensor that can be contructed from metric tensor and its first and second derivatives, that is linear in the second derivatives. Any other combinations simply can't be tensors. This will be important later. Now initially this tensor seems very daunting to deal with. For $d = 4$ dimensions it as $4^4 = 256$ components! That is a lot. But what we can actually notice is that this tensor has a lot of symmetries which will actually bring this number down to 20.

Immediately from the definition: $[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\gamma\mu\nu} V^\gamma$ we can see that since the commutator is anti-symmetric in the μ and ν indices, then we must have that:

$$R^\alpha{}_{\gamma\mu\nu} = -R^\alpha{}_{\gamma\nu\mu} \tag{24}$$

We can also define a tensor with all lower indices by $R_{\delta\gamma\mu\nu} := g_{\delta\alpha} R^\alpha{}_{\gamma\mu\nu}$. With this and a specific choice of co-ordinates it is easy to show that:

$$R_{\delta\gamma\mu\nu} = R_{\mu\nu\delta\gamma} \tag{25}$$

This also implies, through combination with 24 that $R_{\delta\gamma\mu\nu} = -R_{\gamma\delta\mu\nu}$. Through further manipulation we can also find that:

$$R_{\delta\gamma\mu\nu} + R_{\delta\mu\nu\gamma} + R_{\delta\nu\gamma\mu} = 0 \tag{26}$$

This is equivalent to saying $R_{\delta[\gamma\mu\nu]} = 0$. This is known as the **First Bianchi Identity**.

Lastly we can use this relation, along with the fact that the commutator satisfies the Jacobi identity, that is: $[\nabla_\rho, [\nabla_\mu, \nabla_\nu]] + [\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] = 0$ to write down that:

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0 \tag{27}$$

This is the **Second Bianchi Identity**. This relation will be useful later.

We can also construct useful tensors out of the contraction of this 4 −tensor. We can contract one of the upper indices with one of the lower ones in the following way, to get a rank 2 tensor:

$$R_{\mu\nu} := R^\lambda{}_{\mu\lambda\nu}$$

This is known as the **Ricci tensor**. We do the contraction in this specific way as it turns out that $R^\lambda{}_{\lambda\mu\nu}$ is 0 and $R^\lambda{}_{\mu\nu\lambda} = -R^\lambda{}_{\mu\lambda\nu}$. We can also use property 25 to show that this tensor is symmetric, i.e. $R_{\mu\nu} = R_{\nu\mu}$. The other properties also ensure that this is the only rank 2 tensor that we can construct from the curvature tensor as all others vanish.

Furthermore we can contract this tensor with the metric tensor to get a scalar quantity:

$$R := g^{\mu\nu} R_{\mu\nu}$$

This is known as the **Ricci Scalar**. Again by the symmetries above, it turns out this is the only non-vanishing scalar one can construct from the curvature tensor. This quantity has less information about the curvature at each point in space, but it is far more succinct then having 256 numbers at each point.